A LOCAL CHARACTERIZATION OF VC-MINIMALITY

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Abstract. We show VC-minimality is Π_4^0-complete. In particular, we give a local characterization of VC-minimality. We also show dp-smallness is Π_1^1-complete.

1. Introduction

VC-minimality is a notion of simplicity of a first order theory which simultaneously generalizes weak o-minimality and C-minimality. Until now, VC-minimality has been a very difficult notion to work with. This difficulty is due to the complexity of the definition of VC-minimality. In particular, the definition is Σ_1^1, i.e., it requires an existential quantifier over sets of formulae. As such, it is quite difficult to verify that a theory is not VC-minimal. Instead, most instances of proofs that a theory is not VC-minimal actually show that the theory fails to satisfy one of several weaker principles such as convex orderability, dp-smallness, or dp-minimality. In this paper, we answer the following question:

Question 1.1. *How hard is it to determine whether or not a theory is VC-minimal?*

Index sets are a tool used to quantify the complexity of notions. Let P be a property of objects in a class K. Then the index set of P is the set

$$I(P) := \{i \mid i \text{ is an index for a recursive } C \in K \text{ with the property } P\}.$$

By restricting to the recursive $C \in K$, the complexity of this set comes from the complexity of the notion P, not the inherent complexity in the object C.

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The question is formalized as asking to characterize the complexity of the index set I(VC-minimal theories). We show that in fact VC-minimality is far simpler than expected, being Π^0_4-complete, and we give a simple characterization. This characterization makes VC-minimality far easier work with. We hope that it will spur further research in the area. Our characterization, which holds for theories in countable languages, is 'local' in the sense that it gives a condition that must hold for each formula.

We show that the related notion of dp-smallness is, in fact, far more complicated, and is Π^1_1-complete. In this vain, we also answer a question from [6] by giving examples of dp-small theories in countable languages which are not convexly orderable.

We leave the following question open:

Open Question 1.2. *What is the complexity of convex orderability?*

2. Background

Let X be a set, $\mathcal{C} \subseteq \mathcal{P}(X)$. We say $\mathcal{C}$ is *directed* if, for all $A, B \in \mathcal{C}$, at least one of the following holds:

- $A \subseteq B$,
- $B \subseteq A$, or
- $A \cap B = \emptyset$.

For simplicity of notation, for $A, B \subseteq X$, we write $A \perp B$ to denote that $\{A, B\}$ is not directed. That is,

- $A \setminus B \neq \emptyset$,
- $B \setminus A \neq \emptyset$, and
- $A \cap B \neq \emptyset$.

Remark 2.1 (Swiss Cheese Decomposition). Suppose $\mathcal{C} \subseteq \mathcal{P}(X)$ is directed. If $A \in \mathcal{C}$ and $B_i \in \mathcal{C}$ for $i < n$ with $B_i \subseteq A$ for all $i < n$ and $B_i \cap B_j = \emptyset$ for all $i \neq j$, then we call $S = A \setminus (B_0 \cup ... \cup B_{n-1})$ a *swiss cheese*, A is the *wheel* of S and the B_i's are the *holes* of S. If $D \subseteq X$ is a (finite) boolean combination of elements of $\mathcal{C}$, then there exists swiss cheeses $S_0, ..., S_{m-1}$ such that

- $S_i \cap S_j = \emptyset$ for all $i \neq j$,
- no wheel of some S_i is equal to a hole of some S_j, and
- $D = S_0 \cup ... \cup S_{m-1}$.

We call such $S_0, ..., S_{m-1}$ a *swiss cheese decomposition* of D. See Lemma 2.1 of [4] for more details. By Theorem 3.1 of [4], there is a means of canonically choosing a decomposition, so we may consider "the" swiss cheese decomposition of D.

Lemma 2.2 (Union of Chains)**.** *If $\mathcal{C} \subseteq \mathcal{P}(X)$ is directed, $\mathcal{C}_0 \subseteq \mathcal{C}$ is a chain, and $A := \bigcup \mathcal{C}_0$, then $\mathcal{C} \cup \{A\}$ is directed.*

Proof. Fix $B \in \mathcal{C}$. We must show that either $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$. If any $B' \in \mathcal{C}_0$ contains B, then $B \subseteq A$, and we are done. Similarly, if every $B' \in \mathcal{C}_0$ is disjoint from B, then $A \cap B = \emptyset$. The remaining case is where some $B' \in \mathcal{C}_0$ intersects B, but none contains B. Thus this B' is contained in B. As $\mathcal{C}_0$ is a chain, every element intersects B and none contains B, so every member of the chain is contained in B. Thus $A \subseteq B$. □

Definition 2.3. Fix a language L, an L-theory T, and a monster model $\mathcal{U} \models T$.

- We say a set of partitioned L-formulae $\Psi = \{\psi_i(x; y_i) \mid i \in I\}$ is *directed* if the set $\mathcal{C}_\Psi := \{\psi_i(\mathcal{U}; b) \mid i \in I, b \in \mathcal{U}_{y_i}\}$ is directed (in the ambient set $\mathcal{U}_x$).
- We say that the theory T is *VC-minimal* if, there exists a directed set of formulae Ψ (in the free variable x with $|x| = 1$) such that, every (parameter) definable set $A \subseteq \mathcal{U}$ is a (finite) boolean combination of elements from $\mathcal{C}_\Psi$.
- In this case, we call Ψ a *generating family* for T.

If $\mathcal{C} \subseteq \mathcal{P}(X)$ is directed, then $\mathcal{C} \cup \{\{a\} \mid a \in X\}$ is directed. Therefore, without loss of generality, we may assume the formula $x = y$ is in the generating family of any VC-minimal theory.

An L-structure M is called *convexly orderable* if there exists a linear order $\trianglelefteq$ on M (not necessarily definable) such that, for all L-formulas $\varphi(x; y)$ with $|x| = 1$, there exists $k < \omega$ such that, for all $b \in M_y$, $\varphi(M; b)$ is a union of at most k $\trianglelefteq$-convex subsets of M. By Proposition 2.3 of [7], if M is convexly orderable and $N \equiv M$, then N is convexly orderable, so convex orderability is a property of theories.

3. Devastation and Immortality

The following is a technical definition which plays an important role in our local characterization of VC-minimality.

Definition 3.1 (Devastation, Immortality)**.** Suppose that $\psi(x; y)$ is a partitioned L-formula and $\varphi(x)$ is an $L(\mathcal{U})$-formula, both with a common free variable, x. We say that φ *devastates* ψ if there exists a sequence $\langle c_i : i < \omega \rangle$ of elements in $\mathcal{U}_y$ such that, for all $i < j < \omega$,

- $\models \exists x(\psi(x; c_i) \wedge \neg\psi(x; c_j) \wedge \varphi(x))$, and
- $\models \exists x(\psi(x; c_i) \wedge \neg\psi(x; c_j) \wedge \neg\varphi(x))$.

If there exists no $L(\mathcal{U})$-formula $\varphi(x)$ which devastates $\psi(x;y)$, then we say that $\psi(x;y)$ is *immortal*.

Remark 3.2. If $\varphi(x;y)$ is an L-formula such that, for all $d \in \mathcal{U}_y$, $\varphi(x;d)$ does not devastate $\psi(x;z)$, then by compactness there exists $k < \omega$ such that, for all $d \in \mathcal{U}_y$, there does not exist $\langle c_i : i < k \rangle$ from $\mathcal{U}_z$ so that for all $i < j < k$,

- $\models \exists x(\psi(x;c_i) \wedge \neg\psi(x;c_j) \wedge \varphi(x,d))$, and
- $\models \exists x(\psi(x;c_i) \wedge \neg\psi(x;c_j) \wedge \neg\varphi(x,d))$.

It follows that immortality of $\psi(x,y)$ in a recursive theory T is a Π^0_2 condition.

If $\varphi(x)$ devastates $\psi(x;y)$ witnessed by $\langle c_i : i < \omega \rangle$, then by Ramsey's theorem and compactness, we may assume $\langle c_i : i < \omega \rangle$ is indiscernible.

Lemma 3.3 (Directed and Devastated)**.** *Suppose $\varphi(x)$ is an $L(\mathcal{U})$-formula, and $\psi(x;y)$ is a directed L-formula. Then $\varphi(x)$ devastates $\psi(x;y)$ if and only if there exists an indiscernible sequence $\langle c_i : i < \omega \rangle$ in $\mathcal{U}_y$ such that one of the following hold:*

1. *for all $i < \omega$, $\models \forall x(\psi(x;c_{i+1}) \to \psi(x;c_i))$, $\models \exists x(\psi(x;c_i) \wedge \neg\psi(x;c_{i+1}) \wedge \varphi(x))$, and $\models \exists x(\psi(x;c_i) \wedge \neg\psi(x;c_{i+1}) \wedge \neg\varphi(x))$; or*
2. *for all $i < \omega$, $\models \neg\exists x(\psi(x;c_{i+1}) \wedge \psi(x;c_i))$, $\models \exists x(\psi(x;c_i) \wedge \varphi(x))$, and $\models \exists x(\psi(x;c_i) \wedge \neg\varphi(x))$.*

Proof. If (1) or (2) hold, then clearly $\varphi(x)$ devastates $\psi(x;y)$. Conversely, if $\varphi(x)$ devastates $\psi(x;y)$, then by Remark 3.2 we can assume the witness $\langle c_i : i < \omega \rangle$ is indiscernible. Therefore, we have either that, for all $i < \omega$, $\psi(\mathcal{U};c_{i+1}) \subseteq \psi(\mathcal{U};c_i)$ or, for all $i < j < \omega$, $\psi(\mathcal{U};c_i) \cap \psi(\mathcal{U};c_j) = \emptyset$. Now (1) or (2) follow from each case. □

Definition 3.4. If $\varphi(x;y)$ is any formula and a is any parameter, we refer to $\varphi(x;a)$ as an *instance of* φ.

Definition 3.5 (Instance Sums)**.** Fix L-formulae $\varphi(x;y)$ and $\psi(x;z)$. Then their *instance sum* is the following formula

$$(\varphi \oplus \psi)(x;y,z,w_0,w_1) := (w_0 = w_1 \to \varphi(x;y)) \wedge (w_0 \neq w_1 \to \psi(x;z)).$$

Remark 3.6 (On Instance Sums). If $\varphi(x;y)$ and $\psi(x;z)$ are L-formulae, then each instance of $(\varphi \oplus \psi)$ is T-equivalent to either an instance of φ or an instance of ψ. Conversely, each instance of φ and each instance of ψ is T-equivalent to an instance of $(\varphi \oplus \psi)$.

If $\varphi(x;y)$ and $\psi(x;z)$ are immortal L-formulae, then $(\varphi \oplus \psi)$ is immortal. If $\delta(x)$ devastates $(\varphi \oplus \psi)$, then by the pigeonhole principle, either δ devastates φ or δ devastates ψ. This contradicts the assumption that both formulae are immortal.

If $\{\varphi(x;y), \psi(x;z)\}$ is directed, then $(\varphi \oplus \psi)$ is directed.

Lemma 3.7 (Balls are Immortal)**.** *If T is VC-minimal and $\psi(x;z)$ is in the generating family of T, then ψ is immortal.*

Proof. Suppose, by means of contradiction, that $\psi(x;z)$ is in the generating family of T but ψ is not immortal. Therefore, there exists an $L(\mathcal{U})$-formula $\varphi(x)$ which devastates ψ. Then by Lemma 3.3, there are $a_i, b_i \in \mathcal{U}_x$ and $c_i \in \mathcal{U}_z$ such that, for all $i < j < \omega$,

- $a_i \in \varphi(\mathcal{U}) \cap \psi(\mathcal{U}; c_i) \setminus \psi(\mathcal{U}; c_j)$, and
- $b_i \in \neg\varphi(\mathcal{U}) \cap \psi(\mathcal{U}; c_i) \setminus \psi(\mathcal{U}; c_j)$.

Since T is VC-minimal, $\varphi(\mathcal{U})$ has a swiss cheese decomposition, namely $S_0, ..., S_{m-1}$ as in Remark 2.1. Therefore, by the pigeonhole principle, for some $j < m$ we have infinitely many $i < \omega$ such that $a_i \in S_j$. Let $S = S_j$ and, without loss of generality, suppose all $a_i \in S$. Let A be the wheel and $B_0, ..., B_{m-1}$ be the holes of S (if S has no holes, we get a contradiction, since $\psi(\mathcal{U}; c_i) \not\subseteq S$ for any $i < \omega$). By the pigeonhole principle again, there exists $j < m$ and infinitely many $i < \omega$ such that $b_i \in B_j$. Let $B = B_j$. For each $i \geq 1$: since $b_i \in B$, $B \cap \psi(x, c_i) \neq \emptyset$. Since $b_{i-1} \in B$, $B \not\subseteq \psi(x, c_i)$. Thus $\psi(x, c_i) \subseteq B$. But now $a_i \in B$, so $a_i \notin S$, which contradicts our choice of S. $\square$

4. Local characterization of VC-minimality

Theorem 4.1 (Local Characterization of VC-Minimality)**.** *For a theory T in a countable language L, the following are equivalent:*

1. *T is VC-minimal,*
2. *for all L-formulae $\varphi(x;y)$, there exists an immortal directed L-formula $\psi(x;z)$ such that each instance of φ is T-equivalent to a (finite) boolean combination of instances of ψ.*

Since compactness shows that if every instance of φ is equivalent to a boolean combination of instances of ψ, then there is an n so that every instance of φ is a boolean combination of $\leq n$ instances of ψ, this shows that the index set of VC-minimal theories is Π^0_4.

Remark 4.2. Our restriction to a countable language is necessary. Consider the example in the language $L = \{P_i \mid i < \omega_1\}$ with $\aleph_1$-many unary predicates and let T be the L-theory which says that, for all finite disjoint $I, J \subseteq \omega_1$, there are infinitely many x such that

$$\bigwedge_{i \in I} P_i(x) \wedge \bigwedge_{j \in J} \neg P_j(x).$$

This theory has quantifier elimination and is superstable. One can easily check it satisfies condition (2) of Theorem 4.1, but this is not VC-minimal (see Example 2.10 of [7] for more details).

Lemma 4.3 (Main Construction Lemma)**.** *If $\varphi(x; y)$ and $\psi(x; z)$ are each a directed immortal formula (not assuming $\{\varphi, \psi\}$ is directed), then there exists $\delta(x; w)$ an immortal formula such that*

- *$\{\psi, \delta\}$ is directed, and*
- *each instance of φ is a finite boolean combination of instances of ψ and δ.*

As the proof of the Main Construction Lemma is somewhat involved and combinatorial, we leave it to Section 8. We now consider the proof of Theorem 4.1, given the Main Construction Lemma.

Proof of Theorem 4.1. (1) $\Rightarrow$ (2): Suppose T is VC-minimal and fix a L-formula $\varphi(x; y)$. By compactness, there exists a directed family of finitely many L-formulae $\{\psi_i(x; z_i) \mid i < k\}$ such that each instance of φ is T-equivalent to a boolean combination of instances of the ψ_i's. By taking instance sums, we may assume that $k = 1$. By Lemma 3.7, ψ is immortal.

(2) $\Rightarrow$ (1): We construct Ψ the generating family by induction. First, since L is countable, there exists an enumeration $\{\varphi_i(x; y_i) \mid i < \omega\}$ of the L-formulae with x (where $|x| = 1$) as a free variable. Let $\Psi_0 = \emptyset$ and suppose that we have Ψ_i a finite directed set of immortal L-formulae constructed so that, for all $j < i$, each instance of φ_j is T-equivalent to a boolean combination of instances of elements from Ψ_{j+1}. Suppose further that $\Psi_j \subseteq \Psi_{j+1}$ for all $j < i$. Now consider $\varphi_i(x; y_i)$ and let $\psi(x; z)$ be given as in (2) (hence ψ is immortal and directed). Let $\psi'(x; z')$ be the instance sum of Ψ_i, which is immortal and directed by Remark 3.6. By Lemma 4.3, there exists $\delta(x; w)$ an immortal L-formula such that $\{\delta, \psi'\}$ is directed and each instance of ψ is T-equivalent to a boolean combination of instances of ψ' and δ. Therefore, each instance of φ is T-equivalent to a boolean combination of instances of ψ' and δ. Let $\Psi_{i+1} := \Psi_i \cup \{\delta\}$, which is a finite directed set of immortal L-formulae. Finally, let $\Psi = \cup_i \Psi_i$. □

5. Stable VC-Minimal Theories

Lemma 5.1. *Suppose T is VC-minimal and stable. Then, there exists $\Psi := \{E_i(x, y) \mid i \in I\}$ a directed set of equivalence relations (on x with $|x| = 1$) that is a generating family for T.*

Proof. Since T is VC-minimal, let Ψ' be a generating family for T. Now fix $\psi(x;y) \in \Psi'$ and $p \in S_y(\emptyset)$. Suppose, by means of contradiction, that the type

$$p(y_0) \cup p(y_1) \cup \{\psi(\mathcal{U};y_0) \subsetneq \psi(\mathcal{U};y_1)\}$$

is consistent. Take $\langle b_0, b_1 \rangle$ a witness to this and take $\sigma \in \mathrm{Aut}(\mathcal{U})$ sending b_0 to b_1. Let $b_n = \sigma^n(b_0)$ (in particular, this is consistent with the naming of b_1). Then, $\langle b_i : i < \omega \rangle$ and ψ is a witness to the strict order property, a contradiction to the fact that T is stable. Therefore, there exists $\delta(y) \in p(y)$ such that, for all $b_0, b_1 \in \mathcal{U}_{y_i}$ with $\models \delta(b_0) \wedge \delta(b_1)$, either $\psi(\mathcal{U};b_0) = \psi(\mathcal{U};b_1)$ or $\psi(\mathcal{U};b_0) \cap \psi(\mathcal{U};b_1) = \emptyset$. In other words, the formula

$$E_{\psi,p}(x_0,x_1) := (\exists y)(\delta(y) \wedge \psi(x_0;y) \wedge \psi(x_1;y)) \vee (x_0 = x_1)$$

is a $\emptyset$-definable equivalence relation. Now take

$$\Psi := \{E_{\psi,p} \mid \psi(x;y) \in \Psi', p \in S_y(\emptyset)\}.$$

We claim that Ψ is a generating family for T. To show this, we simply show $\mathcal{C}_\Psi = \mathcal{C}_{\Psi'}$. For $A \in \mathcal{C}_{\Psi'}$, $A = \psi(\mathcal{U};b)$ for some $\psi(x;y) \in \Psi'$, $b \in \mathcal{U}_y$. Then, for any $a \in A$, one can check that $A = E_{\psi,\mathrm{tp}(b)}(\mathcal{U};a)$. Conversely, take $A \in \mathcal{C}_\Psi$, so $A = E_{\psi,p}(\mathcal{U},a)$ for some $\psi(x;y) \in \Psi'$, $p \in S(\emptyset)$, and $a \in \mathcal{U}$. Let $\delta(y) \in p(y)$ be the associated formula. If there exists $b \in \mathcal{U}_y$ such that $\models \delta(b) \wedge \psi(a;b)$, then $\psi(\mathcal{U};b) = A$, hence $A \in \mathcal{C}_{\Psi'}$. On the other hand, if there exists no such b, then $A = \{a\}$ so, since $(x = y) \in \Psi'$, $A \in \mathcal{C}_{\Psi'}$. □

So, without loss of generality, when dealing with a VC-minimal stable theory, we may assume the generating family is a set of equivalence relations on the home sort. As a corollary of Theorem 4.1, we get the following characterization of stable VC-minimal theories.

Theorem 5.2. *Suppose T is a stable theory in a countable language. The following are equivalent:*

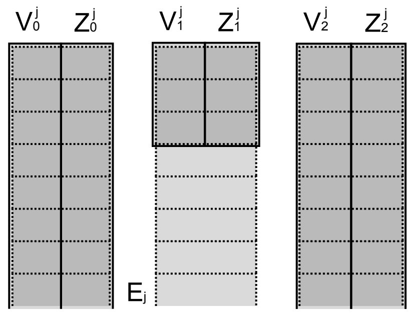
(1) *T is VC-minimal,*
(2) *For each formula $\varphi(x;y)$, there exists finitely many refining definable equivalence relations $\{E_j(x_0,x_1) \mid j < m\}$, each of which is immortal, such that, for all $b \in \mathcal{U}_y$, $\varphi(\mathcal{U};b)$ is a (finite) boolean combination of instances of the E_j's.*

6. Π^0_4-completeness of VC-minimality

We now show that the characterization of VC-minimality given in Theorem 4.1 is the simplest possible.

Theorem 6.1. *The index set of VC-minimal theories is Π^0_4-hard.*

FIGURE 1. Example construction where 1 enters $W_{g(i,j)}$ at stage 3.



Proof. We describe a recursive function f, which, on a given input i, outputs a theory T_i so that T_i is always $\aleph_0$-stable, and T_i is VC-minimal if and only if $i \in S$ for a Π^0_4-complete set S. We have S written as $\forall j(W_{g(i,j)}$ is co-finite) for a fixed recursive function g.

Our theory will be in the language $L := \{E_j \mid j \in \omega\} \cup \{U_j \mid j \in \omega\} \cup \{V^j_k, Z^j_k \mid j,k \in \omega\}$ where each E_j is binary and all other relations are unary.

T_i begins with the following axioms:

- The U_j defines disjoint infinite sets.
- Each E_j is an equivalence relation on U_j with infinitely many infinite classes.
- The V^j_k defines disjoint subsets of U_j.
- If $x \in V^j_k$, $y \in V^j_l$ for $k \neq l$, then $\neg E_j(x,y)$.
- For each $j,k \in \omega$: There are infinitely many E_j-classes which do not intersect V^j_k.
- For each $j,k \in \omega$: For each E_j-class A which intersects V^j_k, both $A \cap V^j_k$ and $A \smallsetminus V^j_k$ are infinite.
- For each $j,k \in \omega$: $x \in Z^j_k$ if and only if $x \notin V^j_k$ and there is a y so that $E_j(x,y) \wedge y \in V^j_k$.

At stage s, for each $k \leq s$, we add the following axioms to T_i:

- If $k \notin W^s_{g(i,j)}$, then add an axiom stating that there are at least s E_j-classes which intersect V^j_k.
- If k enters $W_{g(i,j)}$ at stage s, add an axiom stating that there are exactly s E_j-classes which intersect V^j_k.

See Figure 1 for details.

The following is an easy exercise.

Lemma 6.2. *For every i, T_i is a complete $\aleph_0$-stable theory with quantifier elimination.*

Lemma 6.3. *If $\forall j(W_{g(i,j)}$ is co-finite), then T_i is VC-minimal.*

Proof. For each j, let S_j be the set $\omega \smallsetminus W_{g(i,j)}$. Each S_j is finite, by assumption. Define X_j to be the set of elements in U_j, but not in any V_k^j or Z_k^j for $k \in S_j$. Let Φ be the family composed of the following families of definable sets:

- $\{U_j \mid j \in \omega\}$
- $\{V_k^j, Z_k^j \mid k \in S_j, j \in \omega\}$
- $\{X_j \mid j \in \omega\}$
- $\{E_j(x,y) \wedge x \in V_k^j \mid k \in S_j\}$
- $\{E_j(x,y) \wedge x \in Z_k^j \mid k \in S_j\}$
- $\{E_j(x,y) \wedge x \in X_j\}$
- $\{E_j(x,y) \wedge V_l^j(x) \mid l \notin S_j\}$
- $\{E_j(x,y) \wedge Z_l^j(x) \mid l \notin S_j\}$

It is immediate that Φ is directed. For $l \notin S_j$, V_l^j is a finite union of instances of $\{E_j(x,y) \wedge V_l^j(x)\}$. Similarly for Z_l^j. Each E_j-class is the union of elements of Φ given by the fourth, fifth, and sixth lines. By quantifier elimination, every definable set is a boolean combination of instances from Φ. Thus Φ witnesses VC-minimality of T_i.

□

Lemma 6.4. *If $\exists j(W_{g(i,j)}$ is co-infinite), then T_i is non-VC-minimal.*

Proof. Fix j so $W_{g(i,j)}$ is co-infinite. Let ψ be a directed formula so that every instance of E_j is a boolean combination of instances of ψ. By Lemma 5.1, ψ can be assumed to be comprised of equivalence relations. By quantifier elimination, ψ is comprised of E_j-classes (off of a set defined by finitely many exceptional V_k^j and Z_k^j). Let $k \notin W_{g(i,j)}$ not be one of those finitely many exceptional k. Thus V_k^j intersects infinitely many E_j-classes. This shows that V_k^j devastates ψ. Thus ψ cannot be contained in any family witnessing VC-minimality of T_i by Lemma 3.7, and thus T_i is non-VC-minimal.

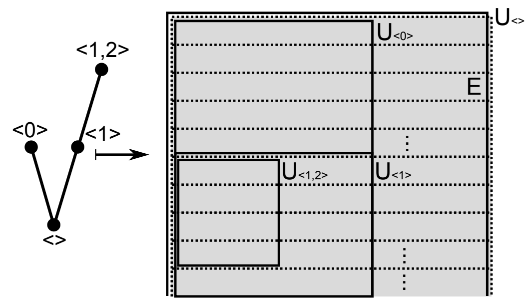
□

□

Corollary 6.5. *The index set of VC-minimal theories is Π_4^0-complete.*

Remark 6.6. One should note that all the theories T_i constructed in Theorem 6.1 are, in fact, convexly orderable. This gives us a large list of examples of theories that are $\aleph_0$-stable and convexly orderable but not VC-minimal.

FIGURE 2. Example construction of a particular tree.



7. COMPLEXITY OF DP-SMALLNESS

Definition 7.1. We say a theory T is *dp-small* if there does not exist an L-formula $\varphi(x;y)$, a sequence $\langle b_i : i < \omega\rangle$, and $L(\mathcal{U})$-formulae $\psi_j(x)$ (where x is of the home sort) such that, for all $i, j < \omega$, the following partial type is consistent with T:

$$\{\varphi(x;b_i), \psi_j(x)\} \cup \{\neg\varphi(x;b_{i'}) \mid i' \neq i\} \cup \{\neg\psi_{j'}(x) \mid j' \neq j\}.$$

A theory being not dp-small is clearly a Σ^1_1 condition, hence the index of dp-small theories is Π^1_1.

Theorem 7.2. *The index set of dp-small theories is* Π^1_1*-complete.*

Proof. We use the fact that $\{T \subset \omega^{<\omega} \mid T \text{ is a recursive tree with no path}\}$ is Π^1_1-complete (see Theorem 5.14 of [3]). Given a (recursive index for a) tree $T \subseteq \omega^{<\omega}$, we produce a theory so that the tree T has a path if and only if the theory is not dp-small. We fix the language $\mathcal{L} := \{E\} \cup \{U_\sigma \mid \sigma \in \omega^{<\omega}\}$ where E is binary and each U_σ is unary.

The theory is axiomatized as follows:

- E is an equivalence relation with infinitely many infinite classes.
- $\forall x U_\emptyset(x)$
- If σ and τ are incomparable, then $U_\sigma \cap U_\tau = \emptyset$.
- If $\sigma \prec \tau$, then $U_\tau \subseteq U_\sigma$.
- If $\tau \notin T$, then $U_\tau = \emptyset$
- If $\tau = \sigma^\frown\langle i\rangle$, and $\tau \in T$, then there is an infinite set S of E-equivalence classes so that for each E-equivalence class $A \in S$, $U_\tau \cap A$ is an infinite co-infinite subset of $U_\sigma \cap A$. Further, there are infinitely many E-equivalence classes which intersect U_σ which do not intersect U_τ.

See Figure 2 for an example.

It is straightforward to verify that the theory produced is complete for any T and is dp-small if and only if T has no infinite path.

□

To conclude this section, we use the ideas behind the construction in Theorem 6.1 to provide an answer to a question from [6].

Example 7.3. We give an example of a theory in a countable language that is dp-small but not convexly orderable, answering a question from [6]. This theory happens to be $\aleph_0$-stable. Let $L = \{E\} \cup \{U_{i,j} \mid j \leq i < \omega\}$, where E is a binary relation and each $U_{i,j}$ is a unary relation. Let T be the L-theory which says

- E is an equivalence relation with infinitely many infinite classes;
- the $U_{i,j}$ are pairwise disjoint;
- for all $i < \omega$, $U_{i,0} \cup ... \cup U_{i,i}$ is a union of infinitely many E-classes;
- if an E-class intersects $U_{i,j}$, it does so with infinitely many points and it intersects each $U_{i,j'}$ for $j' \leq i$; and
- if an E-class intersects $U_{i,j}$, it does not intersect $U_{i',j}$ for $i' \neq i$.

This is $\aleph_0$-stable and has quantifier elimination.

Suppose, by means of contradiction, that it were convexly orderable, say with $\lhd$ on $M \models T$. Then, there exists $k < \omega$ such that, for all $a \in M$, $E(M; a)$ is a union of at most k $\lhd$-convex sets. Look at $U_{2k,j}$ for $j \leq 2k$. Again, by convex orderability, there exists $\ell < \omega$ such that each $U_{2k,j}(M)$ is a union of a most ℓ $\lhd$-convex sets. Let $B_{j,m}$ for $m < \ell$ be the mth $\lhd$-convex component of $U_{2k,j}(M)$ (some may be empty). By the pigeonhole principle, there exists $m_0, ..., m_k < \ell$ and an infinite collection of E-classes $A_0, A_1, ...$ such that $B_{j,m_j} \cap A_t \neq \emptyset$ for all $j \leq k$ and $t < \omega$. As the B_{j,m_j} are $\lhd$-convex and pairwise disjoint, and each intersect A_0 and $M \setminus A_0$, we must have that A_0 is a union of at least $k + 1$ $\lhd$-convex sets. This is a contradiction.

However, this theory is dp-small. Suppose, by means of contradiction, that $\varphi(x; y)$ together with $\psi_\ell(x)$ for $\ell < \omega$ is a witness to non-dp-smallness. That is, there exists $\langle b_i : i < \omega \rangle$ such that, for all $i, \ell < \omega$, the partial type

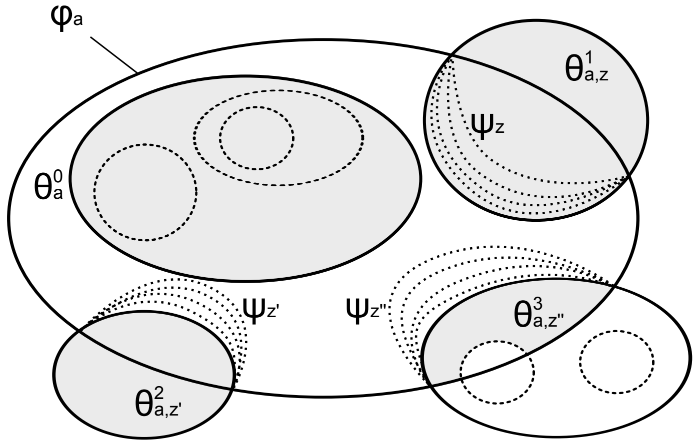
$$\{\varphi(x; b_i), \psi_\ell(x)\} \cup \{\neg\varphi(x; b_{i'}) \mid i' \neq i\} \cup \{\neg\psi_{\ell'}(x) \mid \ell' \neq \ell\}$$

is consistent. By quantifier elimination, we may assume φ is E with perhaps a restriction to some $U_{i,j}$ and that the $\psi_\ell(x)$ are of the form $U_{i,j}$ perhaps restricted to an E-class. One checks such formulae cannot make the above partial type consistent.

8. The Main Construction Lemma

Suppose M is a countable model of a theory T in a countable language. In this seciton, for simplicity of exposition, for a formula $\varphi(x; y)$ and $b \in M_y$, we will write φ_b to mean $\varphi(M; b)$.

FIGURE 3. Example of θ^0, θ^1, θ^2, and θ^3.



Lemma 8.1 (Unions and Intersections of chains). *Suppose $\rho(x;y)$ and $\tau(x;z)$ are so that for any y, z, $\rho_y \not\perp \tau_z$. Let χ be any union of a chain of instances of ρ or intersection of a chain of instances of ρ. Then for every z, $\tau_z \not\perp \chi$.*

Proof. We first suppose χ is a union of a chain of instances of ρ. Suppose z is so that $\tau_z \perp \chi$. Let a be in the intersection and b be in $\chi \setminus \tau_z$. Let ρ_w be in the chain so that it contains a and b. Then $\rho_w \perp \tau_z$, which is a contradiction.

Now suppose χ is an intersection of a chain of instances of ρ. Suppose z is so that $\tau_z \perp \chi$. Let a be any element of $\tau_z \setminus \chi$ and choose w so ρ_w does not contain a. Then $\rho_w \perp \tau_z$, which is a contradiction. □

Lemma 8.2 (Main Construction Lemma). *If $\varphi(x;y)$ and $\psi(x;z)$ are each a directed immortal formula (not assuming $\{\varphi, \psi\}$ is directed), then there exists $\delta(x;w)$ an immortal formula such that*

- *$\{\psi, \delta\}$ is directed, and*
- *each instance of φ is a finite boolean combination of instances of ψ and δ.*

Proof. We begin by defining the following formulae:

- $\theta^0_a := \{x \in \varphi_a \mid \forall z(x \in \psi_z \to \psi_z \not\perp \varphi_a)\}$
- If $\varphi_a \perp \psi_z$, then define $\theta^1_{a,z} := \bigcup\{\psi_y \mid \psi_y \setminus \varphi_a = \psi_z \setminus \varphi_a\}$. Otherwise, $\theta^1_{a,z} := \emptyset$.
- If $\varphi_a \perp \psi_z$, then define $\theta^2_{a,z} := \bigcap\{\psi_y \mid \psi_y \setminus \varphi_a = \psi_z \setminus \varphi_a\}$. Otherwise, $\theta^2_{a,z} := \emptyset$.
- $\theta^3_{a,z} := (\theta^2_{a,z} \cap \varphi_a) \setminus \bigcup\{\psi_{z'} \mid (\psi_{z'} \perp \varphi_a) \wedge \psi_{z'} \cap \varphi_a \subsetneq \theta^2_{a,z} \cap \varphi_a\}$

See Figure 3 for an example.

We intend to show that $\{\psi(x;y), \theta^0(x;a), \theta^1(x;a,z), \theta^2(x;a,z), \theta^3(x;a,z)\}$ is directed, each θ^i is immortal, and that each instance of φ is a boolean combination of instances from this family.

Lemma 8.3. *$\{\psi(x;y), \theta^0(x;a), \theta^1(x;a,z), \theta^2(x;a,z), \theta^3(x;a,z)\}$ is directed*

Proof. For each pair of formulae from $\{\psi(x;y), \theta^0(x;a), \theta^1(x;a,z), \theta^2(x;a,z), \theta^3(x;a,z)\}$, we argue that no two instances can be $\perp$.

ψ, θ^0: If $x \in \psi_y \cap \theta^0_a$, then $\psi_y \not\perp \varphi_a$. If $\psi_y \supset \varphi_a$, then $\theta^0_a \subseteq \varphi_a \subseteq \psi_y$. So we suppose $\psi_y \subseteq \varphi_a$. Take any $x' \in \psi_y$. If x were in some $\psi_{y'}$ where $\psi_{y'} \perp \varphi_a$, then $x \in \psi_y \subseteq \psi_{y'}$, contradicting $x \in \theta^0_a$. Thus $\psi_y \subseteq \theta^0_a$.

ψ, θ^1: This follows from Lemma 8.1 since any instance of θ^1 is a union of a chain of instances of ψ and ψ is directed.

ψ, θ^2: This follows from Lemma 8.1 since any instance of θ^2 is an intersection of a chain of instances of ψ and ψ is directed.

ψ, θ^3: Let ψ_y intersect $\theta^3_{a,z}$. If $\psi_y \supseteq \psi_z \smallsetminus \varphi_a$, then $\psi_y \supseteq \theta^2_{a,z} \supseteq \theta^3_{a,z}$. So, we may assume $\psi_y \smallsetminus \varphi_a \subsetneq \psi_z \smallsetminus \varphi_a$. If for some w, $\psi_w \perp \varphi_a$ and $\psi_w \supseteq \psi_y$ and $\psi_w \subset \theta^2_{a,z}$, then ψ_y is explicitly excluded from $\theta^3_{a,z}$ and the intersection is empty. Otherwise, $\psi_y \subseteq \theta^2_{a,z} \cap \varphi_a$ and it is contained in $\theta^3_{a,z}$.

θ^0, θ^0: We may assume $\varphi_{a'} \subseteq \varphi_a$. If there is no $x \in \theta^0_{a'}$ and y so $x \in \psi_y$ and $\psi_y \perp \varphi_a$, then $\theta^0_{a'} \subseteq \theta^0_a$. Otherwise, this ψ_y must contain $\varphi_{a'}$, since $\psi_y \not\perp \varphi_{a'}$. Thus $\theta^0_{a'} \subseteq \varphi_{a'}$ and $\varphi_{a'} \cap \theta^0_a = \emptyset$.

θ^0, θ^3: Let $x \in \theta^0_a \cap \theta^3_{a',v}$. Let S be the set of v' so that $\psi_v \smallsetminus \varphi_{a'} = \psi_{v'} \smallsetminus \varphi_{a'}$. Then $x \in \psi_{v'}$ for every $v' \in S$. So $\psi_{v'} \not\perp \varphi_a$ for each $v' \in S$. If $\psi_{v'} \subseteq \varphi_a$ for any $v' \in S$, then $\psi_{v'} \subseteq \theta^0_a$ (see ψ, θ^0), so $\theta^3_{a',v} \subseteq \psi_v \subseteq \theta^0_a$. So we assume $\varphi_a \subseteq \psi_{v'}$ for each such v'. Thus $\varphi_a \subseteq \theta^2_{a',v}$. Thus $\varphi_a \subset \varphi_{a'}$. If there is a z so that $\theta^0_a \subseteq \psi_z$ and $\psi_z \perp \varphi_{a'}$ and $\psi_z \cap \varphi_{a'} \subsetneq \theta^2_{a',v} \cap \varphi_{a'}$, then $\theta^0_a \cap \theta^3_{a',v} = \emptyset$. Otherwise, $\theta^0_a \subseteq \theta^3_{a',v}$.

θ^1, anything: Since no instance of ψ is $\perp$ to any instance of a θ^i, this follows by Lemma 8.1.

θ^2, anything: Since no instance of ψ is $\perp$ to any instance of a θ^i, this follows by Lemma 8.1.

$\theta^3_{a,z}, \theta^3_{a',z'}$: As $\theta^3_{a,z} \subseteq \varphi_a$, we may assume $\varphi_a \supseteq \varphi_{a'}$. Similarly, we may assume either $\theta^2_{a,z} \subseteq \theta^2_{a',z'}$ or vice versa. We start with the first case: $\theta^2_{a,z} \subseteq \theta^2_{a',z'}$. Let S be the set of w so that $\psi_w \smallsetminus \varphi_a = \psi_z \smallsetminus \varphi_a$. Then for every $w \in S$, since $\psi_w \perp \varphi_a$, it follows that $\psi_w \perp \varphi_{a'}$. Since $\{\psi, \theta^2\}$ is directed, either ψ_w is a proper subset of $\theta^2_{a',z'}$ and is thus excluded from $\theta^3_{a',z'}$ or ψ_w contains $\theta^2_{a',z'}$. In the first case, $\theta^3_{a,z}$ is disjoint from $\theta^3_{a',z'}$, so we suppose the second case holds for every $w \in S$. Thus $\theta^2_{a,z} \supseteq \theta^2_{a',z'}$. It remains to check that any ψ_y contained in $\theta^2_{a,z}$ excluded from $\theta^3_{a,z}$ is also excluded from $\theta^3_{a',z'}$. If $\psi_y \perp \varphi_a$, then $\psi_y \perp \varphi_{a'}$ and if it defines a proper subset of $\theta^2_{a',z'} \cap \varphi_{a'}$, then it defines a proper subset of $\theta^2_{a,z} \cap \varphi_a$, as needed.

Now we consider the second case: $\theta^2_{a',z'} \subsetneq \theta^2_{a,z}$. If $\theta^2_{a',z'} \cap \varphi_a \subsetneq \theta^2_{a,z} \cap \varphi_a$, then using a small enough instance of $\psi_{w'}$ where $\psi_{w'} \smallsetminus \varphi_{a'} = \psi_{z'} \smallsetminus \varphi_{a'}$, we see that $\theta^2_{a',z'}$ is excluded from $\theta^3_{a,z}$. Thus we may assume $\theta^2_{a',z'} \cap \varphi_a = \theta^2_{a,z} \cap \varphi_a$. It remains to see that any instance of ψ_y omitted from $\theta^3_{a,z}$ is also omitted from $\theta^3_{a',z'}$. Since $\varphi_{a'} \subseteq \varphi_a$, if ψ_y intersects $\theta^2_{a',z'}$ and $\psi_y \perp \varphi_a$, then $\psi_y \perp \varphi_{a'}$. Thus if ψ_y is omitted in the definition of $\theta^3_{a,z}$, it is also omitted in the definition of $\theta^3_{a',z'}$. Thus $\theta^3_{a,z} \subseteq \theta^3_{a',z'}$.

□

Lemma 8.4. *Suppose $\rho(x;y)$ is an immortal formula, and that each instance of $\chi(x;z)$ is a union of a chain of instances of ρ. Then χ is immortal.*

Suppose $\rho(x;y)$ is an immortal formula, and that each instance of $\chi(x;z)$ is an intersection of a chain of instances of ρ. Then χ is immortal.

Proof. First we consider the case where every instance of $\chi(x;z)$ is a union of a chain of instances of ρ. Suppose towards a contradiction that $\gamma(x)$ devastates χ witnessed by the indiscernible $\langle c_i : i < \omega \rangle$. For all $i < j < \omega$, $\chi_{c_i} \smallsetminus \chi_{c_j}$ intersects both γ and $\neg\gamma$. Since each χ instance is a union of a chain of instances of ρ, there exists d_i for each $i < \omega$ so that $\rho_{d_i} \subseteq \chi_{c_i}$ and for all $i < j$, $\rho_{d_i} \smallsetminus \chi_{c_j}$ intersects both γ and $\neg\gamma$. This witnesses that γ devastates ρ, contrary to the assumption of ρ's immortality.

Now we consider the case where every instance of $\chi(x;z)$ is an intersection of a chain of instances of ρ. Suppose towards a contradiction that $\gamma(x)$ devastates χ witnessed by the indiscernible $\langle c_i : i < \omega \rangle$. For all $i < j < \omega$, $\chi_{c_i} \smallsetminus \chi_{c_j}$ intersects both γ and $\neg\gamma$. Since each χ instance is an intersection of a chain of instances of ρ, there exists d_i for each $i < \omega$ so that $\chi_{c_i} \subseteq \rho_{d_i}$ and for all $i < j$, $\rho_{d_i} \smallsetminus \rho_{d_j}$ intersects both γ and $\neg\gamma$. This witnesses again that γ devastates ρ, contrary to the assumption of ρ's immortality.

□

Lemma 8.5. *θ^0 is immortal.*

Proof. Towards a contradiction, suppose γ devastates θ^0 and consider the indiscernible sequence $\langle a_i : i < \omega \rangle$ witnessing this as in Lemma 3.3. If $\varphi_{a_0} \cap \varphi_{a_1} = \emptyset$, then γ devastates φ by indiscernibility, contradicting the immortality of φ. If $\varphi_{a_0} \subseteq \varphi_{a_1}$, then one of two cases holds:

(i) There exists z such that $\varphi_{a_0} \subseteq \psi_z$ and $\psi_z \perp \varphi_{a_1}$. In this case, $\theta^0_{a_1} \cap \varphi_{a_0} = \emptyset$, hence $\theta^0_{a_0}$ and $\theta^0_{a_1}$ are disjoint and $\theta^0_{a_1} \subseteq (\varphi_{a_1} \setminus \varphi_{a_0})$. Therefore, by indiscernibility, γ devastates φ, contrary to assumption.

(ii) There exists no such z. Then $\varphi_{a_0} \subseteq \theta^0_{a_1}$, hence $\theta^0_{a_0} \subseteq \theta^0_{a_1}$, but this contradicts the choice of the a_i's in Lemma 3.3.

Similarly, if $\varphi_{a_1} \subseteq \varphi_{a_0}$ and there exists z such that $\varphi_{a_1} \subseteq \psi_z$ and $\psi_z \perp \varphi_{a_0}$, then this contradicts the immortality of φ. Therefore, we must have that $\varphi_{a_1} \subseteq \varphi_{a_0}$ and no such z exists. Hence $\theta^0_{a_1} \subseteq \theta^0_{a_0}$.

As γ does not devastate φ, we must have that $(\varphi_{a_0} \setminus \varphi_{a_1})$ is contained in either γ or $\neg\gamma$. Without loss of generality, suppose it is contained in γ. Then, by indiscernibility, $(\varphi_{a_i} \setminus \varphi_{a_{i+1}}) \subseteq \gamma$ for all $i < \omega$.

Notice that $\neg\gamma \cap (\theta^0_{a_0} \setminus \theta^0_{a_1}) \neq \emptyset$ by assumption, so choose x in this set. As $x \notin \theta^0_{a_1}$, there exists z such that $x \in \psi_z$ and $\psi_z \perp \varphi_{a_1}$, hence $\psi_z \cap \theta^0_{a_1} = \emptyset$. However, since $x \in \theta^0_{a_0}$ and $\{\psi, \theta^0\}$ is directed, we must have that $\psi_z \subseteq \theta^0_{a_0}$. Therefore, $\psi_z \subseteq (\theta^0_{a_0} \setminus \theta^0_{a_1})$. Moreover, as $\psi_z \perp \varphi_{a_1}$ and $\psi_z \subseteq \varphi_{a_0}$, we have that $\psi_z \cap (\varphi_{a_0} \setminus \varphi_{a_1}) \neq \emptyset$. Hence, $\psi_z \cap \gamma \neq \emptyset$. By indiscernibility, there are z_i such that

- $\psi_{z_i} \cap \neg\gamma \neq \emptyset$,
- $\psi_{z_i} \cap \gamma \neq \emptyset$, and
- $\psi_{z_i} \subseteq (\theta^0_{a_i} \setminus \theta^0_{a_{i+1}})$.

In particular, the ψ_{z_i}'s are disjoint. Hence, γ devastates ψ, contrary to immortality of ψ.

□

Lemma 8.6. *θ^1 is immortal*

Proof. This follows from Lemma 8.4. □

Lemma 8.7. *θ^2 is immortal*

Proof. This follows from Lemma 8.4. □

Lemma 8.8. *θ^3 is immortal.*

Proof. For this proof, let

$$\theta^4_{a,b,z} := (\theta^2_{a,z} \cap \varphi_b) \smallsetminus \bigcup \{\psi_{z'} \mid (\psi_{z'} \perp \varphi_b) \wedge \psi_{z'} \cap \varphi_b \subsetneq \theta^2_{a,z} \cap \varphi_b\}.$$

In particular, $\theta^4_{a,a,z} = \theta^3_{a,z}$, so it suffices to show θ^4 is immortal.

By means of contradiction, suppose γ devastates θ^4, and consider the indiscernible sequence $\langle\langle a_i, b_i, z_i\rangle : i < \omega\rangle$ witnessing this as in Lemma 3.3. Fix any $i \neq j$. Since θ_2 is directed, we have three cases:

(i) $\theta^2_{a_i,z_i} \cap \theta^2_{a_j,z_j} = \emptyset$,
(ii) $\theta^2_{a_i,z_i} \subseteq \theta^2_{a_j,z_j}$, or
(iii) $\theta^2_{a_j,z_j} \subseteq \theta^2_{a_i,z_i}$.

For Case (i), since γ devastates θ^4 and $\theta^4_{a_i,b_i,z_i} \subseteq \theta^2_{a_i,z_i}$, we have that γ devastates θ^2 by indiscernibility. Case (ii) and (iii) are symmetric, so let us suppose that Case (ii) holds. In almost the exact same way as one shows that $\psi_y \not\perp \theta^3_{a,z}$ for any a, y, z, one can show that $\psi_y \not\perp \theta^4_{a,b,z}$ for any a, b, y, z. Hence, $\theta^2_{c,y} \not\perp \theta^4_{a,b,z}$ for any a, b, c, y, z by Lemma 8.1. Thus, there are three subcases:

(a) $\theta^2_{a_i,z_i} \cap \theta^4_{a_j,b_j,z_j} = \emptyset$,
(b) $\theta^2_{a_i,z_i} \subseteq \theta^4_{a_j,b_j,z_j}$, or
(c) $\theta^4_{a_j,b_j,z_j} \subseteq \theta^2_{a_i,z_i}$.

In Case (a), $(\theta^4_{a_j,b_j,z_j} \setminus \theta^4_{a_i,b_i,z_i}) \subseteq (\theta^2_{a_j,z_j} \setminus \theta^2_{a_i,z_i})$, therefore γ devastates θ^2 by indiscernibility. In Case (b), fix $k < i < j$ or $k > i > j$. Then, by indiscernibility, $\theta^2_{a_k,d_k} \subseteq \theta^4_{a_i,b_i,z_i}$ and, by definition, $\theta^4_{a_j,b_j,z_j} \subseteq \theta^2_{a_j,z_j}$. Hence,

$$(\theta^4_{a_j,b_j,z_j} \setminus \theta^4_{a_i,b_i,z_i}) \subseteq (\theta^2_{a_j,z_j} \setminus \theta^2_{a_k,d_k}).$$

Therefore, γ devastates θ^2 by indiscernibility. Hence Case (c) must hold. Together, (ii) and (c) imply $\theta^4_{a_i,b_j,z_i} = \theta^4_{a_j,b_j,z_j}$. Hence, by indiscernibility, we may assume there are a and z such that, for all $i < \omega$, $\theta^4_{a,b_i,z} = \theta^4_{a_i,b_i,z_i}$. We now consider the sequence $\langle\langle a, b_i, z\rangle : i < \omega\rangle$ which witnesses that θ^4 is devastated by γ.

If $\varphi_{b_0} \cap \varphi_{b_1} = \emptyset$, then, as $\theta^4_{a,b_i,z} \subseteq \varphi_{b_i}$ for all i, γ devastates φ. So we may assume that $\varphi_{b_1} \subseteq \varphi_{b_0}$ (note that, if $\varphi_{b_0} \subseteq \varphi_{b_1}$, then $\theta^4_{a,b_0,z} \subseteq \theta^4_{a,b_1,z}$, contrary to this sequence witnessing devastation of θ^4). If both γ and $\neg\gamma$ intersect $\varphi_{b_0} \setminus \varphi_{b_1}$, then γ devastates φ. So, without loss of generality (and by indiscernibility), we may assume $\varphi_{b_i} \setminus \varphi_{b_{i+1}} \subseteq \gamma$ for all $i < \omega$. In particular, note that $\neg\gamma$ must intersect $\bigcap_{i<\omega} \varphi_{b_i}$.

Since $\neg\gamma$ intersects $\theta^4_{a,b_0,z} \setminus \theta^4_{a,b_1,z}$, there exists w such that

- ψ_w intersects $\neg\gamma$,
- $\psi_w \subseteq \theta^2_{a,z}$,
- $\psi_w \perp \varphi_{b_1}$, and
- $\psi_w \subseteq \varphi_{b_0}$.

In particular, ψ_w intersects $(\varphi_{b_0} \setminus \varphi_{b_1})$, hence also γ. For all $i < \omega$, ψ_w does not contain $(\varphi_{b_i} \setminus \varphi_{b_{i+1}})$ as otherwise $\theta^4_{a,b_i,z} = \theta^4_{a,b_{i+1},z}$, contrary to the choice of b_i. On the other hand, for all but finitely many i, ψ_w

does not intersect $(\varphi_{b_i} \setminus \varphi_{b_{i+1}})$, as otherwise ψ_w would devastate φ. By removing finitely many and reindexing, we may assume ψ_w is disjoint from $(\varphi_{b_i} \setminus \varphi_{b_{i+1}})$ for all $i > 1$.

By indiscernibility, for each $i < \omega$, there exists w_i such that

- ψ_{w_i} intersects γ and $\neg\gamma$,
- ψ_{w_i} intersects $(\varphi_{b_{2i}} \setminus \varphi_{b_{2i+1}})$, and
- ψ_{w_i} is disjoint from $(\varphi_{b_{2k}} \setminus \varphi_{b_{2k+1}})$ for all $k \neq i$

(the last condition is clear for $k > i$ and, for $k < i$, note that $\psi_{w_i} \subseteq \varphi_{b_{2i}}$, hence ψ_{w_i} is disjoint from $(\varphi_{b_{2k}} \setminus \varphi_{b_{2k+1}})$). In particular, since ψ is directed, the last two conditions imply that the ψ_{w_i}'s are disjoint. Hence, by the first condition, γ devastates ψ, contrary to immortality of ψ.

□

Lemma 8.9. *For any c, φ_c is a boolean combination of instances from $\{\psi, \theta^0, \theta^1, \theta^2, \theta^3\}$.*

Proof. Every element in φ_c is either in θ^0_c or is in some $\theta^1_{c,z}$. We first note that there is a finite set of instances of $\theta^1_{c,z}$ which suffices to cover $(\varphi_c \smallsetminus \theta^0_c)$. Otherwise, we could choose more and more instances of $\theta^1_{c,z}$ which would witness that φ_c devastates θ^1.

We now define a sequence of sets whose union will be φ_c. Set $Y_0 = \theta^0_c$. Suppose we have defined the sets Y_j for $j < i$. Suppose further that there is a finite set S_{i-1} of elements so that $(\bigcup_{w \in S_{i-1}} \theta^1_{c,w} \cap \varphi_c) = (\varphi_c \smallsetminus \bigcup_{j<i} Y_j)$. Now we define

$$Y_i := \bigcup_{w \in S_{i-1}} ((\theta^1_{c,w} \smallsetminus \theta^2_{c,w}) \cup \theta^3_{c,w}).$$

To complete the recursive definition of the sequence of sets Y_i for $i < \omega$, we need to see that there is a finite set S_i so that $(\bigcup_{w \in S_i} \theta^1_{c,w} \cap \varphi_c) = (\varphi_c \smallsetminus \bigcup_{j \leq i} Y_j)$. We build S_i as follows: Having selected elements $a_0, \dots a_{k-1}$ so that $(\bigcup_{j<k} \theta^1_{c,a_j} \cap \varphi_c) \subsetneq (\varphi_c \smallsetminus \bigcup_{j \leq i} Y_j)$, we need to select an element a_k. Fix an element $x \in \varphi_c \smallsetminus (\bigcup_{j \leq i} Y_j \cup \bigcup_{j<k} \theta^1_{c,a_j})$ and let a_k be an element so $x \in \theta^1_{c,a_k}$. By directedness of $\{\theta^0, \theta^1, \theta^3\}$, $\theta^1_{c,a_k} \subseteq \varphi_c \smallsetminus (\bigcup_{j \leq i} Y_j \cup \bigcup_{j<k} \theta^1_{c,a_j})$. This process must stop, yielding a finite set S_i, as otherwise φ_c devastates θ^1.

It remains to see that for some i, $\bigcup_{j \leq i} Y_j = \varphi_c$. Otherwise there is an infinite sequence of b_i for $i \in \omega$ so that $b_i \in S_i$ for each i and $\theta^1_{c,b_{i+1}} \subseteq \theta^1_{c,b_i}$, and this sequence witnesses that φ devastates θ^1.

As each Y_j is a boolean combination of instances from $\{\psi, \theta^0, \theta^1, \theta^2, \theta^3\}$, φ_c is a boolean combination of instances from $\{\psi, \theta^0, \theta^1, \theta^2, \theta^3\}$. □

□

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