# CONVEXLY ORDERABLE GROUPS AND VALUED FIELDS

JOSEPH FLENNER AND VINCENT GUINGONA

Abstract. We consider the model theoretic notion of convex orderability, which fits strictly between the notions of VC-minimality and dp-minimality. In some classes of algebraic theories, however, we show that convex orderability and VC-minimality are equivalent, and use this to give a complete classification of VC-minimal theories of ordered groups and abelian groups. Consequences for fields are also considered, including a necessary condition for a theory of valued fields to be quasi-VC-minimal. For example, the $p$-adics are not quasi-VC-minimal.

## 1. Introduction

After many of the advancements in modern stability theory, some model theorists have been seeking to adapt techniques from stable model theory to other families of unstable, yet still well-behaved theories. These include o-minimal theories as well as theories without the independence property. As these notions of model-theoretic tameness proliferate, in each case, two natural questions arise: what are the useful consequences of the property, and which interesting theories have the property? As an example of the latter line of inquiry, an ordered group is weakly o-minimal if and only if it is abelian and divisible, and an ordered field is weakly o-minimal if and only if it is real closed [10]. Similar characterizations of dp-minimality for abelian groups can be found in [3], and results on dp-minimal ordered groups can be found in [13].

Resting comfortably among these conditions is VC-minimality, introduced by Adler in [2]. Most of the classical variations on minimality, such as (weak) o-minimality, strong minimality, and C-minimality, imply VC-minimality. On the other hand, VC-minimality is strong enough to imply many properties of recent interest, such as dependence and dp-minimality.

The question of consequences of VC-minimality has been addressed elsewhere (see e.g. [4, 7, 8]). In this paper, we seek to identify the VC-minimal theories among some basic classes of algebraic structures. Here a problem quickly arises. While it tends to be straightforward to verify that a theory is VC-minimal, the definition of VC-minimality does not lend itself easily to negative results. Except in some special cases, previously it had only been possible to show a theory is not VC-minimal by showing that it is not dp-minimal or dependent.

To sidestep this problem, we explore the intermediate notion of convex orderability, first introduced in [8]. All VC-minimal theories are also convexly orderable,

*Date*: July 7, 2013.

2010 *Mathematics Subject Classification.* Primary: 03C60. Secondary: 20A05, 06F15, 12J10.

*Key words and phrases.* Convexly orderable, VC-minimality, ordered groups, valued fields, abelian groups.

Both authors were supported by NSF grant DMS-0838506.

and while the converse fails in general, in many cases it is, in a sense, close enough. The strategy, thus, is twofold. Given a class of algebraic theories, we use known results (for example, on o-minimal ordered groups) to produce a list of VC-minimal theories from the class. We then study convex orderability in relation to the class of theories to establish that the list is exhaustive.

In this way, we give a complete classification of VC-minimal theories of ordered groups (Section 3) and abelian groups (Section 5). Partial results, in the form of necessary conditions for VC-minimality, are given for ordered fields (Section 3) and valued fields (Section 4). For valued fields, the weaker condition of quasi-VC-minimality is also evaluated.

The remainder of this section gives the necessary background on VC-minimality, and Section 2 presents some useful facts about convex orderability.

1.1. **VC-minimality.** Let $X$ be any set and let $\mathcal{B} \subseteq \mathcal{P}(X)$. We say that $\mathcal{B}$ is *directed* if, for all $A, B \in \mathcal{B}$, one of the following conditions holds:

(1) $A \subseteq B$,
(2) $B \subseteq A$, or
(3) $A \cap B = \emptyset$.

Let $T$ be a first-order $\mathcal{L}$-theory, and fix a set of formulas

$$\Psi = \{\psi_i(x; \bar{y}_i) \mid i \in I\}$$

(note that the singleton $x$ is a free variable in every formula of $\Psi$, but the parameter variables $\bar{y}_i$ may vary). Then $\Psi$ is *directed* if, for all $\mathfrak{M} \models T$,

$$\left\{\psi_i(\mathfrak{M}; \bar{a}) \;\middle|\; i \in I, \bar{a} \in M^{|\bar{y}_i|}\right\}$$

is directed, where $\psi_i(\mathfrak{M}; \bar{a}) = \{b \in M \mid \mathfrak{M} \models \psi_i(b; \bar{a})\} \subseteq M$.

We say that $T$ is *VC-minimal* if there exists a directed $\Psi$ such that all (parameter-definable) formulas $\varphi(x)$ are $T$-equivalent to a boolean combination of instances of formulas from $\Psi$ (i.e., formulas of the form $\psi(x; \bar{a})$ for $\psi \in \Psi$). In this case, $\Psi$ is called a *generating family* for $T$.

For example, it is easy to see that strongly minimal theories are VC-minimal; take $\Psi = \{x = y\}$. Similarly, o-minimal theories are VC-minimal; take $\Psi = \{x \leq y, x = y\}$. A prototypical example of a VC-minimal theory which is neither stable nor o-minimal is the theory of algebraically closed valued fields; take $\Psi = \{v(z) < v(x - y), v(z) \leq v(x - y)\}$, recalling the swiss cheese decomposition of Holly [9]. By a simple type-counting argument, one can see that formulas $\varphi(x; \bar{y})$ in VC-minimal theories have VC-density $\leq 1$ (see [3]). From this, one can conclude that VC-minimal theories are dp-minimal (see, for instance, [6]).

Finally, $T$ is *quasi-VC-minimal* if there exists a directed $\Psi$ such that all formulas $\varphi(x)$ are $T$-equivalent to a boolean combination of instances of formulas from $\Psi$ and parameter-free formulas. Clearly, all VC-minimal theories are quasi-VC-minimal. Moreover, the theory of Presburger arithmetic, $\mathrm{Th}(\mathbb{Z}; +, \leq)$, is quasi-VC-minimal; take $\Psi = \{x \leq y, x = y\}$. Again, by the same type-counting argument, one can check that quasi-VC-minimal theories are dp-minimal.

## 2. Convex orderability

VC-minimality is a powerful condition having many consequences (see, for example, [2, 4, 7, 8]). However, it can be difficult to verify that a theory is not VC-minimal. In attempting to classify VC-minimal theories of certain kinds, therefore, we instead look at a related notion called convex orderability.

**Definition 2.1.** An $\mathcal{L}$-structure $\mathfrak{M}$ is *convexly orderable* if there exists a linear order $\trianglelefteq$ on $M$ (not necessarily definable) such that, for all $\varphi(x;\bar{y})$, there exists $k < \omega$ such that, for all $\bar{b} \in M^{|\bar{y}|}$, $\varphi(\mathfrak{M};\bar{b})$ is a union of at most $k$ $\trianglelefteq$-convex subsets of $M$.

Note in the above that $k$ may depend on $\varphi$, but $\trianglelefteq$ does not. In [8], it is shown that if $\mathfrak{M}$ is convexly orderable and $\mathfrak{M} \equiv \mathfrak{N}$, then $\mathfrak{N}$ is convexly orderable as well. Therefore, convex orderability is a property of a theory. Moreover, the next proposition follows immediately from the definition.

**Proposition 2.2.** *The property of convex orderability is closed under reducts. That is, if $T$ is a convexly orderable $\mathcal{L}$-theory and $\mathcal{L}' \subseteq \mathcal{L}$, then the reduct $T \upharpoonright \mathcal{L}'$ is also convexly orderable.*

For later reference, we cite the following from [8].

**Proposition 2.3** (Corollary 2.9 of [8])**.** *If $T$ is convexly orderable, then $T$ is dp-minimal.*

Furthermore, the following proposition is a simple modification of Proposition 2.5 of [8].

**Proposition 2.4.** *Suppose $X$ is a set and $\mathcal{B} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ is directed. Then, there exists a linear ordering $\trianglelefteq$ on $X$ so that every $B \in \mathcal{B}$ is a $\trianglelefteq$-convex subset of $X$.*

From this, a simple compactness argument gives the corollary.

**Corollary 2.5** (Theorem 2.4 of [8])**.** *If $T$ is VC-minimal and $\mathfrak{M} \models T$, then $\mathfrak{M}$ is convexly orderable.*

By contrast, the above corollary does not hold for quasi-VC-minimal theories, as the $\emptyset$-definable sets may be quite complicated. However, restricting our attention to a single formula, we obtain a localized result for quasi-VC-minimal theories. In the following, notice that $\trianglelefteq$ *does* depend on the formula $\varphi$.

**Corollary 2.6.** *If $T$ is a quasi-VC-minimal theory, $\mathfrak{M} \models T$, and $\varphi(x;\bar{y})$ is a formula, then there exists a linear ordering $\trianglelefteq$ on $M$ and $k < \omega$ such that, for all $\bar{b} \in M^{|\bar{y}|}$, $\varphi(\mathfrak{M};\bar{b})$ is a union of at most $k$ $\trianglelefteq$-convex subsets of $M$. That is, $T$ is 'locally convexly orderable'.*

*Proof.* By compactness, there exists $k_0 < \omega$, $\delta(x;\bar{z})$ a directed formula, and a $\emptyset$-definable partition of $M$ via the finite set of formulas $\Theta(x)$ so that, for each $\bar{b} \in M^{|\bar{y}|}$, $\varphi(\mathfrak{M};\bar{b})$ is a boolean combination of at most $k_0$ instances of $\delta$ and formulas from $\Theta$. (More precisely, compactness yields $k_0$ and a finite set of formulas, while coding tricks allow one to compress a finite set of directed formulas into the single formula $\delta$.)

Let $k = k_0|\Theta| + 1$ and, for each $\theta \in \Theta$, let $\delta_\theta(x;\bar{z})$ be the formula $\delta(x;\bar{z}) \wedge \theta(x)$. Note that each $\delta_\theta$ is directed, as $\delta$ is. Hence, by Theorem 2.4, for each $\theta \in \Theta$, there exists $\trianglelefteq_\theta$ a linear ordering on $\theta(\mathfrak{M})$ so that every instance of $\delta_\theta$ is $\trianglelefteq_\theta$-convex. We then concatenate the orderings $\trianglelefteq_\theta$ in an arbitrary (but fixed) sequence to form a single linear ordering $\trianglelefteq$ on $M$.

Now, for any $\overline{b} \in M^{|\overline{y}|}$ and $\theta \in \Theta$, $\varphi(x;\overline{b}) \wedge \theta(x)$ is a boolean combination of at most $k_0$ instances of $\delta_\theta$, each of which is $\trianglelefteq$-convex. Therefore, $\varphi(\mathfrak{M};\overline{b})$ is a union of at most $k = k_0|\Theta| + 1$ $\trianglelefteq$-convex subsets of $M$. □

One of the original motives for defining convex orderability was to give an analog to VC-minimality which is closed under reducts. However, the converse to Corollary 2.5 does not hold. The dense circle order is convexly orderable but not VC-minimal (for more information, see [2]). It is, in fact, a reduct of (a definitional expansion of) dense linear orders without endpoints, which is o-minimal and hence VC-minimal. On the other hand, the dense circle order becomes VC-minimal if one allows a single parameter in the generating family.

Let us call a theory *VC-minimal with parameters* if there exists a directed generating family as in the original definition, but allowing parameters from some distinguished model in the formulas. One could then ask whether VC-minimality with parameters is closed under reducts. An example in [1] shows that this is still not the case. Recalling Proposition 2.2, therefore, there are convexly orderable theories which are not VC-minimal even with parameters.

Nevertheless, in the following sections we will see several instances where convex orderability serves as a useful proxy for VC-minimality. In particular, we use Corollaries 2.5 and 2.6 to answer questions about which algebraic structures of various kinds are convexly orderable, VC-minimal, and quasi-VC-minimal.

## 3. Ordered groups

Let $\mathfrak{G} = (G; \cdot, \leq)$ be an infinite ordered group and let $T = \mathrm{Th}(\mathfrak{G})$. We prove the following theorem.

**Theorem 3.1.** *The following are equivalent:*

*(1) $\mathfrak{G}$ is abelian and divisible.*
*(2) $T$ is o-minimal,*
*(3) $T$ is VC-minimal,*
*(4) $T$ is convexly orderable.*

This is a generalization of Theorem 5.1 of [10], which is itself a generalization of Theorem 2.1 of [11]. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are well-known (or clear from the previous section), so it will suffice to show that (4) ⇒ (1).

Thus, suppose that $T$ is convexly orderable. By Proposition 3.3 of [13], all dp-minimal ordered groups are abelian. Using Proposition 2.3, therefore, we already have that $\mathfrak{G}$ is abelian and it remains only to show that it is divisible. We begin with a general lemma about convexly orderable ordered structures.

**Lemma 3.2.** *If $\mathfrak{M} = (M; \leq, ...)$ is a linearly ordered structure that is convexly orderable, then there do not exist definable sets $X_0, X_1, ... \subseteq M$ that are pairwise disjoint and coterminal (that is, cofinal or coinitial) in $M$.*

*Proof.* Suppose that $\mathfrak{M}$ is convexly ordered by $\trianglelefteq$. Suppose that there exists definable sets $X_0, X_1, ... \subseteq M$ that are pairwise disjoint and $\leq$-coterminal in $M$. By the pigeonhole principle, we may assume that all $X_i$ are either $\leq$-cofinal or $\leq$-coinitial in $M$. Without loss of generality, suppose all are $\leq$-cofinal in $M$. By convex orderability, for each $i$, $X_i$ is a union of finitely many $\trianglelefteq$-convex subsets of $M$. Therefore, there exists some $\trianglelefteq$-convex subset $C_i \subseteq X_i$ such that $C_i$ is $\leq$-cofinal in $M$.

Because the rays $[a,\infty)_\leq$ are uniformly definable, there is a natural number $k$ such that every $[a,\infty)_\leq$ is the union of at most $k$ $\trianglelefteq$-convex sets. Now consider the sets $C_1,\dots,C_{2k+1}$. Since these are $\trianglelefteq$-convex and pairwise disjoint, we may arrange the indices so that

$$C_{i_1} \lhd C_{i_2} \lhd \ldots \lhd C_{i_{2k+1}}.$$

For each $j \leq 2k+1$, choose $b_j \in C_{i_j}$, and fix $a > \max\{b_j \mid 1 \leq j \leq 2k+1\}$. By $\leq$-cofinality of $C_{i_j}$, for each $j$ we may also choose $c_j \in C_{i_j} \cap [a,\infty)_\leq$. Thus we have

$$c_1 \lhd b_2 \lhd c_3 \lhd \ldots \lhd b_{2k} \lhd c_{2k+1}$$

with each $c_j \in [a,\infty)_\leq$ and each $b_j \notin [a,\infty)_\leq$. It follows that for $j = 0,\dots,k$, each $c_{2j+1}$ lies in a separate $\trianglelefteq$-convex component of $[a,\infty)_\leq$. This contradiction implies that $\mathfrak{M}$ is not convexly orderable, as required. □

We return to the case of $T = \mathrm{Th}(\mathfrak{G})$, where $\mathfrak{G} = (G;+,\leq)$ is a convexly orderable ordered group. For $k < \omega$, let $k \mid x$ be the formula $\exists y\,(k \cdot y = x)$. For each natural number $n \geq 1$ and prime $p$, define the set

$$D_{p,n} = \left\{x \in G \;\middle|\; x > 0, p^n \mid x \text{ and } p^{n+1} \nmid x\right\}.$$

**Lemma 3.3.** *Suppose for some prime $p$ that $pG \neq G$. Then for each $n$, $D_{p,n}$ is cofinal in $G$.*

*Proof.* Since $pG \neq G$, there is some $c > 0$ with $p \nmid c$. Consider $0 < a \in G$. We show that there is $x \geq a$ such that $x \in D_{p,n}$. First, if $p \nmid a$, let $b = a$; if $p \mid a$, set $b = a + c$. So, $b \geq a$ and $p \nmid b$. Now $x = p^n \cdot b \geq a$ and $x \in D_{p,n}$. □

Combining this with Lemma 3.2, we can now easily establish Theorem 3.1.

**Corollary 3.4.** *If $\mathfrak{G}$ is convexly orderable, then $\mathfrak{G}$ is divisible.*

*Proof.* Suppose $\mathfrak{G}$ is convexly orderable but not divisible, say $pG \neq G$. For each $n$, $D_{p,n}$ is cofinal and pairwise disjoint in $\mathfrak{G}$. Apply Lemma 3.2 to conclude. □

Although there were previously known examples of dp-minimal theories that are not VC-minimal (e.g., see [6]), this gives us a natural example of such a theory (discovered independently in [1]).

*Example* 3.5. The theory of Presburger arithmetic, $T = \mathrm{Th}(\mathbb{Z};+,\leq)$, is not VC-minimal and not convexly orderable. On the other hand, it is quasi-VC-minimal, and hence also dp-minimal.

This has interesting consequences for ordered fields.

**Proposition 3.6.** *Suppose $\mathfrak{F} = (F;+,\cdot,\leq)$ is an ordered field. If $\mathfrak{F}$ is convexly orderable, then every positive element has an $n^{th}$ root for all $n \geq 1$.*

*Proof.* Suppose $\mathfrak{F}$ is convexly ordered by $\trianglelefteq$. Then, $\trianglelefteq$ induces a convex ordering on the ordered group $(F_+;\cdot,\leq)$ where $F_+ = \{a \in F \mid a > 0\}$. Thus, by Theorem 3.1, $F_+$ is divisible. In other words, for any $a \in F_+$ and $n \geq 1$, there exists $b \in F_+$ such that $b^n = a$. □

Theorem 5.3 of [10] states that any weakly o-minimal ordered field is real closed. This suggests the following open question.

**Open Question 3.7.** *Is it the case that an ordered field $(F; +, \cdot, \leq)$ is convexly orderable if and only if $(F; +, \cdot, \leq)$ is real closed?*

Before we get carried away, however, not all ordered structures that are convexly orderable are weakly o-minimal. For example, consider $\mathbb{Q}$ and take $D \subseteq \mathbb{Q}$ dense and codense. One can verify that the structure $\mathfrak{M} = (\mathbb{Q}; \leq, D)$ has quantifier elimination, from which it easily follows that it is VC-minimal. For instance, take as a generating family

$$\Psi = \{(D(x) \wedge x < y), (\neg D(x) \wedge x < y), D(x), x = y\}.$$

So $\mathfrak{M}$ is convexly orderable, but on the other hand, $\mathfrak{M}$ is clearly not weakly o-minimal. The issue is that Lemma 3.2 necessitates *infinitely many* coterminal disjoint sets to contradict convex orderability. This leads to another open question.

**Open Question 3.8.** *If $\mathfrak{M} = (M; \leq, ...)$ is a linearly ordered structure that is convexly orderable, then is $\mathfrak{M}$ quasi-weakly o-minimal?*

## 4. Valued fields

4.1. **Simple interpretability.** In this subsection we exhibit a means of passing convex orderability from a structure to a simple interpretation in the structure. If $\mathfrak{M}$ and $\mathfrak{N}$ are models (not necessarily in the same language) and $A \subseteq M$, then $\mathfrak{M}$ *interprets* $\mathfrak{N}$ *over* $A$ if there are $n \geq 1$, an $A$-definable subset $S \subseteq M^n$, and an $A$-definable equivalence relation $\varepsilon$ on $S$ such that

- the elements of $\mathfrak{N}$ are in bijection with the $\varepsilon$-equivalence classes of $S$, and
- the relations on $S$ induced by the relations and functions of $\mathfrak{N}$ via this bijection are $A$-definable in $\mathfrak{M}$.

Moreover, if $n = 1$ in the above definition, we say that $\mathfrak{M}$ *simply interprets* $\mathfrak{N}$.

It is generally most convenient to identify the elements of $\mathfrak{N}$ with the equivalence classes of $S$, so that for instance we will write $\bar{a} \in x$ if $\bar{a} \in S$ and $x \in N$ corresponds to the $\varepsilon$-equivalence class containing $\bar{a}$.

*Remark* 4.1. Using the same notation as above, suppose $\varphi(\bar{x}; \bar{y})$ is a formula in the language of $\mathfrak{N}$ with $k = |\bar{x}|$. Then there is $\tilde{\varphi}(\bar{z}; \bar{w})$ in the language of $\mathfrak{M}$ (with parameters from $A$) with the property that, for any set $X \subseteq N^k$ defined by an instance $\varphi(\bar{x}; \bar{a})$ of $\varphi$, the set

$$\tilde{X} = \bigcup X \subseteq S^k.$$

is defined by an instance $\tilde{\varphi}(\bar{z}; \bar{b})$ of $\tilde{\varphi}$. To see this, induct on the complexity of $\varphi$, replacing function and relation symbols from $\mathfrak{N}$ with their corresponding definitions in $\mathfrak{M}$ and $=$ with $\varepsilon$, and relativizing all quantifiers to $S$.

**Lemma 4.2.** *If $\mathfrak{M}$ simply interprets $\mathfrak{N}$ and $\mathfrak{M}$ is convexly orderable, then $\mathfrak{N}$ is also convexly orderable.*

*Proof.* Let $\varepsilon(x, y)$ define an equivalence relation on $S \subseteq M$ as in the definition of interpretation (possibly over parameters), and suppose that $\mathfrak{M}$ is convexly ordered by $\trianglelefteq_M$. Define on $\mathfrak{N}$ the relation $\trianglelefteq_N$ by

$$x \trianglelefteq_N y \iff (\forall s \in y)(\exists r \in x)[r \trianglelefteq_M s].$$

We claim that $\mathfrak{N}$ is convexly ordered by $\trianglelefteq_N$.

First note that $\trianglelefteq_N$ linearly orders $N$. Transitivity and linearity are clear. For antisymmetry, suppose that $x \trianglelefteq_N y$ and $y \trianglelefteq_N x$. Then, beginning with an arbitrary $s_0 \in y$, find $r_i \in x$, $s_i \in y$ such that for every $i < \omega$, $r_i \trianglelefteq_M s_i$ and $s_{i+1} \trianglelefteq_M r_i$. But since $x$ is a definable subset of $\mathfrak{M}$, $x$ must be a finite union of $\trianglelefteq_M$-convex sets. So we must have $s_i \in x$ for some $i$, whence $x = y$. A similar argument shows that $x \triangleleft_N y$ iff there is an $r \in x$ such that $r \triangleleft_M s$ for all $s \in y$.

Now consider a formula $\varphi(x; \bar{y})$ in the language of $\mathfrak{N}$, $\bar{a}$ a tuple from $N$, and $X \subseteq N$ the set defined by $\varphi(x; \bar{a})$. For $\tilde{\varphi}(x; \bar{b})$ defining $\tilde{X}$ as in Remark 4.1, since $\trianglelefteq_M$ convexly orders $\mathfrak{M}$, there is a uniform bound $k$ on the number of $\trianglelefteq_M$-convex sets comprising an instance of $\tilde{\varphi}$ in $\mathfrak{M}$. It will suffice to show that $X$ is also a union of at most $k$ $\trianglelefteq_N$-convex sets in $N$.

Suppose not, so that there are

$$c_0 \trianglelefteq_N c_1 \trianglelefteq_N \ldots \trianglelefteq_N c_{2k}$$

such that $c_i \in X$ iff $i$ is even. For each $i < 2k$, since $c_i \neq c_{i+1}$ there is $\tilde{c}_i \in c_i$ such that $\tilde{c}_i \triangleleft_M d$ for all $d \in c_{i+1}$. Take also any element $\tilde{c}_{2k} \in c_{2k}$. Now

$$\tilde{c}_0 \trianglelefteq_M \tilde{c}_1 \trianglelefteq_M \ldots \trianglelefteq_M \tilde{c}_{2k}$$

and $\tilde{c}_i \in \tilde{X}$ iff $i$ is even. This contradicts the fact that $\tilde{X}$ is a union of $k$ (or fewer) $\trianglelefteq_M$-convex sets.

We conclude that in $\mathfrak{N}$, every instance of $\varphi$ defines a union of $k$ or fewer $\trianglelefteq_N$-convex sets. Since any formula in the language of $\mathfrak{N}$ admits such a uniform bound, $\trianglelefteq_N$ convexly orders $\mathfrak{N}$. □

Lemma 4.2 allows us to show that a theory is not convexly orderable (hence not VC-minimal) by simply interpreting a structure that is not convexly orderable. We can apply this to theories of valued fields. Let $K$ be a valued field with value group $\Gamma$, residue field $k$, and valuation $v : K \to \Gamma \cup \{\infty\}$, and let $T = \mathrm{Th}(K; +, \cdot, |)$. Here $x|y$ means $v(x) \leq v(y)$. Though we work in the one-sorted language $\mathcal{L} = \{+, \cdot, |\}$, the statements could be adapted to other languages of valued fields.

**Corollary 4.3.** *If $T$ is convexly orderable, then both the value group $\Gamma$ and the residue field $k$ are convexly orderable.*

*Proof.* Both $\Gamma$ and $k$ are simply interpretable (over $\emptyset$) in $K$. For example, $\Gamma$ is interpreted on $S = K \setminus \{0\}$ via $\varepsilon(x, y) \equiv x \mid y \wedge y \mid x$ (i.e., $v(x) = v(y)$). Since $v(xy) = v(x) + v(y)$, the addition in $\Gamma$ is interpreted by multiplication in $K$, and the ordering is explicitly given by $|$. We use Lemma 4.2 to conclude. □

We know that the theory of algebraically closed valued fields is convexly orderable. Also, the theory of real closed valued fields is weakly o-minimal [5], hence also convexly orderable. This leads to an interesting open question: Under which circumstances does the converse of Corollary 4.3 hold?

**Open Question 4.4.** *Is it true that, for any Henselian valued field $K$ with value group $\Gamma$ and residue field $k$, $K$ is convexly orderable if and only if $\Gamma$ and $k$ are convexly orderable?*

We understand when $\Gamma$ is convexly orderable by Theorem 3.1, but we do not currently have a characterization for when $k$ is convexly orderable. Answering Open Question 4.4 would probably require first understanding when a field is convexly orderable in general.

We can apply Corollary 4.3 to the case of the $p$-adics.

**Corollary 4.5.** *If $\Gamma$ is not divisible, then $T$ is not convexly orderable, hence not VC-minimal. In particular, the theory of the $p$-adics is not VC-minimal.*

*Proof.* By Theorem 3.1, $\Gamma$ is convexly orderable if and only if $\Gamma$ is divisible. Hence, if $\Gamma$ is not divisible, then Corollary 4.3 implies that $T$ is not convexly orderable. In particular, the theory of the $p$-adics, $\mathrm{Th}(\mathbb{Q}_p; +, \cdot, |)$, has value group $(\mathbb{Z}; +, \leq)$, which is not divisible. Hence, the theory of the $p$-adics is not VC-minimal. □

By Section 6 of [6], the theory of the $p$-adics is dp-minimal. So this corollary gives us another natural example of a theory that is dp-minimal but not VC-minimal. In the next subsection, we exhibit a means of producing examples of theories that are dp-minimal but not quasi-VC-minimal.

4.2. **Quasi-VC-minimality.** For this subsection, fix $K$ a valued field with value group $\Gamma$ and let $T = \mathrm{Th}(K; +, \cdot, |)$ as in the previous subsection. First, recall that if $K$ is algebraically closed, then $T$ is VC-minimal. Notice that if $K$ is algebraically closed, then $\Gamma$ is divisible. The main goal of this section is to prove the following stronger result.

**Theorem 4.6.** *If $T$ is quasi-VC-minimal, then $\Gamma$ is divisible.*

Suppose then that $\Gamma$ is not divisible, say $p\Gamma \neq \Gamma$. Fix some positive $\gamma_1 \in \Gamma \setminus p\Gamma$. Define $\gamma_n \in \Gamma$ by

$$\gamma_n = \begin{cases} k \cdot p \cdot \gamma_1 & \text{if } n = 2k, \\ \gamma_1 + k \cdot p \cdot \gamma_1 & \text{if } n = 2k+1. \end{cases}$$

Notice that $0 = \gamma_0 < \gamma_1 < ... < \gamma_n < ...$ and $p \mid \gamma_n$ if and only if $n$ is even.

We now construct, for each $n < \omega$, $\mathcal{A}_n \subseteq K$ as follows. Set $\mathcal{A}_0 = \{0\}$. For each $a \in \mathcal{A}_n$, choose $a' \in K$ such that $v(a - a') = \gamma_n$. Let

$$\mathcal{A}_{n+1} = \mathcal{A}_n \cup \{a' \mid a \in \mathcal{A}_n\}.$$

Note that $a' \notin \mathcal{A}_n$ (to see this, show inductively that for distinct $b_1, b_2 \in \mathcal{A}_n$, $v(b_1 - b_2) \leq \gamma_{n-1}$). Therefore $|\mathcal{A}_n| = 2^n$. Moreover, for all $a \in \mathcal{A}_n$ and all $i < n$, there exists $b \in \mathcal{A}_n$ such that $v(a - b) = \gamma_i$.

Suppose that $\trianglelefteq$ is a linear ordering on $K$. In this case, each $\mathcal{A}_n$ is also linearly ordered by $\trianglelefteq$. For each $b \in K$, define

$$X_b = \{a \in K \mid p \mid v(a - b)\}.$$

**Lemma 4.7.** *For each $n < \omega$, there exists $b \in K$ such that $X_b$ is the union of no fewer than $n + 1$ $\trianglelefteq$-convex subsets of $K$.*

*Proof.* Fix $n < \omega$ and let $\mathcal{A} = \mathcal{A}_{2n+1}$, which is a finite linear order (under $\trianglelefteq$).

Let $a_0 \in \mathcal{A}$ be the $\trianglelefteq$-minimal element. In general, we inductively construct a sequence $a_0, ..., a_{2n+1} \in \mathcal{A}$ such that

1. $v(a_j - a_i) = \gamma_j$ for all $j < i$,
2. $a_0 \triangleleft a_1 \triangleleft ... \triangleleft a_{2n+1}$, and
3. for all $a \in \mathcal{A}$ with $v(a - a_i) \geq \gamma_i$, $a_i \trianglelefteq a$.

Suppose that $a_0, \ldots, a_i$ with the above properties have been found, and choose $a_{i+1} \in \mathcal{A}$ $\trianglelefteq$-minimal such that $v(a_{i+1} - a_i) = \gamma_i$. This exists by definition of $\mathcal{A} = \mathcal{A}_{2n+1}$. By condition (3), $a_i \triangleleft a_{i+1}$, so condition (2) holds up to $a_{i+1}$. Condition (1) and $v(a_{i+1} - a_i) = \gamma_i > \gamma_j$ implies that $v(a_j - a_{i+1}) = \gamma_j$ for all $j < i$. Therefore, condition (1) holds for $a_{i+1}$. Finally, fix $a \in \mathcal{A}$ and suppose $v(a - a_{i+1}) \geq \gamma_{i+1}$.

Since $v(a_{i+1} - a_i) = \gamma_i$, we have $v(a - a_i) = \gamma_i$ as well. However, since $a_{i+1}$ was chosen $\trianglelefteq$-minimal in the set $\{x \in \mathcal{A} \mid v(x - a_i) = \gamma_i\}$ and $a$ belongs to this set, we must have that $a_{i+1} \trianglelefteq a$. Thus, condition (3) holds for $a_{i+1}$.

Finally, set $b = a_{2n+1}$. Then, for $i \leq 2n$, $a_i \in X_b$ if and only if $p \mid v(a_i - b)$ if and only if $p \mid \gamma_i$. Recall, moreover, that $p \mid \gamma_i$ if and only if $i$ is even. Therefore, $a_i \in X_b$ if and only if $i$ is even. By condition (2), $X_b$ is the union of no fewer than $n + 1$ $\trianglelefteq$-convex subsets of $K$. □

*Proof of Theorem 4.6.* Suppose $\Gamma \neq p\Gamma$. Fix the formula

$$\varphi(x; y) = \exists z (z^p \mid (x - y)).$$

Towards a contradiction, suppose $T$ were quasi-VC-minimal. By Corollary 2.6, there exists a linear order $\trianglelefteq$ on $K$ and $n < \omega$ such that each instance of $\varphi$ is a union of at most $n$ $\trianglelefteq$-convex subsets of $K$. By Lemma 4.7, there exists $b \in K$ such that $X_b = \varphi(K; b)$ is a union of no fewer than $n+1$ $\trianglelefteq$-convex subsets of $K$, a contradiction. □

**Corollary 4.8.** *The following theories are not quasi-VC-minimal:* $\mathrm{Th}(\mathbb{Q}_p; +, \cdot, |)$ *for any prime* $p$*, and* $\mathrm{Th}(k((t)); +, \cdot, |)$ *for any field* $k$*.*

Since the $p$-adics are dp-minimal, this gives us a natural example of a theory that is dp-minimal and not quasi-VC-minimal. Combining this observation with Corollary 3.5, we get strict implications

$$\text{VC-minimal} \;\Rightarrow\; \text{quasi-VC-minimal} \;\Rightarrow\; \text{dp-minimal}$$

where strictness is witnessed by Presburger arithmetic and the $p$-adics respectively.

## 5. Abelian Groups

Let $\mathfrak{A} = (A; +)$ be an abelian group and $T = \mathrm{Th}(\mathfrak{A})$. Throughout this section we work exclusively in the pure group language $\mathcal{L} = \{+\}$. For each $k, m < \omega$, consider the formula

$$\varphi_{k,m}(x) = \exists y \left(k \cdot y = m \cdot x\right).$$

Notice that $\varphi_{k,m}(\mathfrak{A})$ is a subgroup of $A$. For $k = 0$, $\varphi_{0,m}(\mathfrak{A})$ is the subgroup of $m$-torsion elements of $A$, which we will also denote by $A[m]$. For $m = 1$, $\varphi_{k,1}(\mathfrak{A})$ is the subgroup of $k$-multiples of $A$, which we will also denote by $kA$.

**Proposition 5.1** (Corollary 2.13 of [12])**.** *All definable subsets of $A$ are boolean combinations of cosets of $\varphi_{k,m}(\mathfrak{A})$ for various $k, m < \omega$.*

Let $\mathrm{PP}(A)$ be the set of all the p.p.-definable subgroups of $A$, which are namely the finite intersections of subgroups of the form $\varphi_{k,m}(\mathfrak{A})$ for various $k, m < \omega$. Define a quasi-order $\precsim$ on all subgroups of $A$ by setting, for each subgroup $B_0$ and $B_1$ of $A$:

$$B_0 \precsim B_1 \text{ if and only if } [B_0 : B_0 \cap B_1] < \aleph_0.$$

Think of this as $B_0$ being almost a subgroup of $B_1$ (missing only by a finite index). This quasi-order generates an equivalence relation $\sim$, which is called *commensurability.* For any $B_0 \sim B_1$, notice that $B_0 \cap B_1 \sim B_0$, so $\sim$-classes are closed under intersection. We denote by $\widetilde{\mathrm{PP}}(A)$ the set $\mathrm{PP}(A)/\sim$ of equivalence classes. Thus, $\precsim$ induces a partial order on $\widetilde{\mathrm{PP}}(A)$. In [3], this partial order is used to characterize dp-minimality of $T$ as follows.

**Proposition 5.2** (Corollary 4.12 of [3]). *The theory $T$ is dp-minimal if and only if $\left(\widetilde{\mathrm{PP}}(A); \precsim\right)$ is linear.*

This is then used as the main tool for proving a classification of dp-minimal theories of abelian groups. In the following, a *nonsingular* group $B$ is one for which $B[p]$ and $B/pB$ are finite for all primes $p$.

**Proposition 5.3** (Proposition 5.27 of [3]). *The theory $T$ is dp-minimal if and only if $\mathfrak{A}$ is elementarily equivalent to one of the following abelian groups:*

*(1)* $\bigoplus_{i\geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_i)} \oplus \mathbb{Z}\left(p^\infty\right)^{(\beta)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma)} \oplus B$ *for some prime $p$, a nonsingular abelian group $B$, and $\alpha_i$, $\beta$, and $\gamma$ cardinals with $\alpha_i < \aleph_0$ for all $i$.*

*(2)* $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(\alpha)} \oplus \left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\beta)} \oplus B$ *for some prime $p$, $k \geq 1$, finite abelian group $B$, and cardinals $\alpha$ and $\beta$, at least one of which is infinite.*

In this section, we will prove a characterization for when $T$ is VC-minimal (and convexly orderable) analogous to Proposition 5.2, and likewise use it to obtain a complete list of VC-minimal theories of abelian groups.

**Lemma 5.4.** *Suppose that there exists $\mathcal{H} \subseteq \mathrm{PP}(A)$ such that*

*(1) $(\mathcal{H}; \subseteq)$ is a linear order; and*

*(2) For all $k$ and $m$, $\varphi_{k,m}(\mathfrak{A})$ is a boolean combination of cosets of elements $H \in \mathcal{H}$.*

*Then, $T$ is VC-minimal.*

*Proof.* For each $H \in \mathcal{H}$, let $\psi_H(x; y)$ be the formula $x - y \in H$, and let $\Psi = \{\psi_H \mid H \in \mathcal{H}\}$. The instances of $\Psi$ define precisely the cosets of members of $\mathcal{H}$. We claim that $\Psi$ is a generating family for $T$.

First, to see that $\Psi$ is directed, fix $H_1, H_2 \in \mathcal{H}$ and $a_1, a_2 \in A$. By (1), we may assume without loss of generality that $H_1 \subseteq H_2$. Then each coset of $H_1$ is a subset of a coset of $H_2$, so that either $a_1 + H_1 \subseteq a_2 + H_2$ or $(a_1 + H_1) \cap (a_2 + H_2) = \emptyset$ as required.

By Proposition 5.1, all definable subsets of $A$ are boolean combinations of cosets of $\varphi_{k,m}(\mathfrak{A})$ for various $k, m < \omega$. So (2) implies that all parameter-definable subsets of $A$ are in fact boolean combination of cosets of elements $H \in \mathcal{H}$. □

**Corollary 5.5.** *The theory $T = \mathrm{Th}(\mathbb{Z}; +)$ is VC-minimal.*

*Proof.* Let $\mathcal{H} = \{(n!) \cdot \mathbb{Z} \mid 1 \leq n < \omega\} \cup \{0\}$. This satisfies the conditions in Lemma 5.4. □

For a prime $p$, let $\mathbb{Z}_{(p)}$ be the additive group of the ring $\mathbb{Z}$ localized at the prime ideal $(p) = p\mathbb{Z}$. Let $\mathbb{Z}(p^\infty)$ be the Prüfer $p$-group, which is the direct limit of $(\mathbb{Z}/p^k\mathbb{Z})$ for all $k \geq 1$. For an abelian group $A$ and cardinal $\kappa$, let $A^{(\kappa)}$ be the direct sum of $\kappa$ copies of $A$.

**Corollary 5.6.** *The theories of the following abelian groups are VC-minimal:*

*(1)* $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)}$ *for some $k < \omega$ and prime $p$,*

*(2)* $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)} \oplus \left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\aleph_0)}$ *for some $k < \omega$ and prime $p$, and*

*(3)* $\mathbb{Z}(p^\infty)^{(\beta)} \oplus \mathbb{Z}_{(p)}^{(\gamma)}$ *for cardinals $\beta$ and $\gamma$ and prime $p$.*

*Proof.* (1) Since $p^iA = \left(p^i\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)}$ and $A[p^i] = \left(p^{k-i}\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)}$, we see that

$$\mathrm{PP}(A) = \left\{ \left(p^i\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)} \;\middle|\; 0 \leq i \leq k \right\},$$

which is itself a chain. We conclude that $T$ is VC-minimal by Lemma 5.4.

(2) Notice that, for each $i$,

$$p^iA = \left(p^i\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)} \oplus \left(p^i\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\aleph_0)}$$
$$A[p^i] = \left(p^{k-i}\mathbb{Z}/p^k\mathbb{Z}\right)^{(\aleph_0)} \oplus \left(p^{k+1-i}\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\aleph_0)}.$$

So, let $\mathcal{H}$ be the chain $0 \subset p^kA \subset A[p] \subset p^{k-1}A \subset A[p^2] \subset ...$ and use Lemma 5.4 to conclude.

(3) In this case, we have

$$A[p^i] = (\mathbb{Z}(p^\infty)[p^i])^{(\beta)} \oplus 0$$
$$p^iA = \mathbb{Z}(p^\infty)^{(\beta)} \oplus \left(p^i\mathbb{Z}_{(p)}\right)^{(\gamma)}.$$

Use the chain $0 \subseteq A[p] \subseteq A[p^2] \subseteq ... \subseteq p^2A \subseteq pA \subseteq A$ along with Lemma 5.4 to conclude. □

However, not every dp-minimal abelian group is VC-minimal or even convexly orderable.

**Lemma 5.7.** *Suppose that there exists a chain of $\emptyset$-definable subgroups $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq ...$ and a $\emptyset$-definable subgroup $B \subseteq A$ such that*

*(1) for each $i < \omega$, $A_i \cap B \neq A_{i+1} \cap B$, and*

*(2) for each $i < \omega$, $[A_i : A_i \cap B] \geq \aleph_0$.*

*Then, $T = \mathrm{Th}(A; +)$ is not convexly orderable. Hence, $T$ is not VC-minimal.*

*Proof.* By way of contradiction, suppose that $\mathfrak{A}$ is convexly ordered by $\trianglelefteq$. In particular, suppose that each instance of the formula $x - y \in B$ is a union of at most $k$ $\trianglelefteq$-convex subsets of $A$ for some fixed $k < \omega$.

Since $[A_k : A_k \cap B] \geq \aleph_0$, the set

$$\mathcal{C} = \{a + B \mid a \in A_k\}$$

of cosets of $B$ is infinite. On the other hand, for each $1 \leq i \leq k$, $A_i \cap B \subsetneq A_{i-1} \cap B$. So, for any choice of $b \in (A_{i-1} \setminus A_i) \cap B$ and $a \in A_k$, $a + b \in (A_{i-1} \setminus A_i)$. Therefore, for all $a \in A_k$ and $1 \leq i \leq k$, $(a+B) \cap (A_{i-1} \setminus A_i)$ is non-empty. That is, $(A_{i-1} \setminus A_i)$ intersects non-trivially each element of $\mathcal{C}$.

By convex orderability, for each $i \leq k$, $A_i$ is a finite union of $\trianglelefteq$-convex subsets of $A$. Let $\mathcal{C}_i$ denote the elements $a + B \in \mathcal{C}$ such that, for some $\trianglelefteq$-convex component $C$ of $a + B$, $C \nsubseteq A_i$ and $C \cap A_i \neq \emptyset$. By convexity, there can be only finitely many such $a + B$, namely the ones covering the finitely many "endpoints" of $A_i$. Hence, $\mathcal{C}_i$ is finite for each $i \leq k$. Finally, set

$$\mathcal{C}^* = \mathcal{C} \setminus \left( \bigcup_{i \leq k} \mathcal{C}_i \right).$$

Since $\mathcal{C}$ is infinite, $\mathcal{C}^*$ is also infinite and, in particular, non-empty.

We claim that each $A_i$ contains at most $k - i$ $\trianglelefteq$-convex components of each element of $\mathcal{C}^*$. By choice of $k$, this clearly holds for $i = 0$. So suppose that $i > 0$ and that the claim holds for $A_{i-1}$. Consider $a + B \in \mathcal{C}^*$. By construction, for each

$\trianglelefteq$-convex component $C$ of $a+B$, either $C \subseteq A_i$ or $C \cap A_i = \emptyset$. However, as observed above $(a+B) \cap (A_{i-1} \setminus A_i) \neq \emptyset$, so at least one of the $\trianglelefteq$-convex components of $a+B$ contained in $A_{i-1}$ must be disjoint from $A_i$. By assumption, $A_{i-1}$ contains at most $k-(i-1)$ $\trianglelefteq$-convex components of $a+B$. Thus $A_i$ contains at most $k-i$. The conclusion follows by induction.

Therefore, for all $a+B \in \mathcal{C}^*$, $(a+B) \cap A_k = \emptyset$. On the other hand, $A_k$ intersects every coset $a+B \in \mathcal{C}$ by definition of $\mathcal{C}$. This gives the desired contradiction. □

We use this to produce an example of an abelian group whose theory is dp-minimal but not VC-minimal.

**Corollary 5.8.** *Fix some $\alpha_i < \aleph_0$ for each $i \geq 1$ such that the set $\{i \mid \alpha_i > 0\}$ is infinite. Then the theory of the abelian group*

$$A = \bigoplus_{i \geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{\alpha_i}$$

*is not convexly orderable.*

*Proof.* Let $I = \{i \mid \alpha_i > 0\}$, let $i_0 = 0$, and let $i_1 < i_2 < \ldots$ enumerate $I$. It is straightforward to check that the $\emptyset$-definable subgroups

$$A_\ell = p^{i_\ell} A \text{ for all } \ell < \omega, \text{ and } B = A[p]$$

satisfy the hypotheses of Lemma 5.7. □

By Proposition 5.3 (1), we see that this $A$ is, in fact, dp-minimal.

**Definition 5.9.** For $X \in \widetilde{\mathrm{PP}}(A)$ (i.e., $X$ is a $\sim$-class of $\mathrm{PP}(A)$), we say that $X$ is *upwardly coherent* if there exists $H \in X$ such that, for all $H_1 \in \mathrm{PP}(A)$ with $H \precsim_{\not\sim} H_1$, we have that $H \subseteq H_1$.

By extension, we say that the group $A$ is *upwardly coherent* if every $X \in \widetilde{\mathrm{PP}}(A)$ is.

Intuitively, upward coherence means the class contains a particular subgroup for which being *almost* a proper subgroup is sufficient to be, in fact, a subgroup. In the presence of dp-minimality, this condition implies VC-minimality as shown in the next lemma.

**Lemma 5.10.** *Suppose $T = \mathrm{Th}(A; +)$ is dp-minimal. If $A$ is upwardly coherent, then $T$ is VC-minimal.*

*Proof.* For each $X \in \widetilde{\mathrm{PP}}(A)$, let $H_X \in X$ witness that $X$ is upwardly coherent. Since $\mathrm{PP}(A)$ is countable, so is $X$, so let $X = \{H_i \mid i < \omega\}$ enumerate $X$. Define $H_X^i \in X$ inductively as follows:

- $H_X^0 = H_X$.
- For $i \geq 0$, $H_X^{i+1} = H_X^i \cap H_i$.

Since $X$ is closed under intersection, each $H_X^i$ is still an element of $X$.

Let $\mathcal{H}_X = \left\{H_X^i \;\middle|\; i < \omega\right\}$. By construction, $\mathcal{H}_X$ is a chain under $\subseteq$ with maximal element $H_X$. Moreover, by definition of $\sim$, every $H \in X$ is a *finite* union of cosets of a member of $\mathcal{H}_X$. Finally, set

$$\mathcal{H} = \bigcup \left\{\mathcal{H}_X \;\middle|\; X \in \widetilde{\mathrm{PP}}(A)\right\}.$$

For any distinct $X, Y \in \widetilde{\mathrm{PP}}(A)$, by Proposition 5.2 either $X \precsim\!\!\!\!\not\sim Y$ or $Y \precsim\!\!\!\!\not\sim X$. Without loss, suppose $X \precsim\!\!\!\!\not\sim Y$. Therefore, by upward coherence, $H \supseteq H_X$ for all $H \in Y$. Hence, $\mathcal{H}_X \cup \mathcal{H}_Y$ is a chain under $\subseteq$. It follows that $\mathcal{H}$ is itself a chain under $\subseteq$. We thus conclude that $\mathcal{H}$ satisfies the hypotheses of Lemma 5.4, showing that $T$ is VC-minimal. ☐

Putting this all together, we arrive at the desired characterization of convexly orderable (and VC-minimal) abelian groups.

**Theorem 5.11.** *The following are equivalent:*

*(1) $T$ is VC-minimal;*
*(2) $T$ is convexly orderable;*
*(3) $T$ is dp-minimal and $A$ is upwardly coherent.*

*Proof.* We have (1) $\Rightarrow$ (2) by Corollary 2.5. Lemma 5.10 gives (3) $\Rightarrow$ (1). Thus, it remains only to show (2) $\Rightarrow$ (3).

If $T$ is convexly orderable, then $T$ is dp-minimal by Proposition 2.3. So, suppose that there exists some $X \in \widetilde{\mathrm{PP}}(A)$ that is not upwardly coherent. Fixing any $B \in X$, we construct $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ from $\mathrm{PP}(A)$ such that, for all $i < \omega$:

(1) $[B : A_i \cap B] < \aleph_0$, so that $A_i \cap B \in X$;
(2) If $i > 0$, then $A_{i-1} \cap B \neq A_i \cap B$; and
(3) $[A_i : A_i \cap B] \geq \aleph_0$.

By Lemma 5.7, this implies that $T$ is not convexly orderable, as required.

First, set $A_0 = A$. If $B \sim A$, then $A \in X$ trivially witnesses upward coherence, contrary to assumption. Therefore, $[A : B] \geq \aleph_0$, giving condition (3) for $i = 0$. Clearly condition (1) and (2) also hold for $i = 0$.

Now fix $i \geq 0$ and suppose that $A_i$ has been constructed satisfying (1), (2), and (3). Consider $A_i \cap B$. Since $A_i \cap B \in X$ and $X$ is not upwardly coherent, there exists $H \in \mathrm{PP}(A)$ such that $A_i \cap B \precsim\!\!\!\!\not\sim H$ and $A_i \cap B \not\subseteq H$. Set $A_{i+1} = H \cap A_i$. We show that $A_{i+1}$ satisfies (1), (2), and (3).

Since $A_i \cap B \precsim H$,

$$[A_i \cap B : A_{i+1} \cap B] = [A_i \cap B : H \cap A_i \cap B] < \aleph_0,$$

giving condition (1). Suppose $A_i \cap B = A_{i+1} \cap B$. Then $H \cap (A_i \cap B) = A_i \cap B$ implies $(A_i \cap B) \subseteq H$, contrary to assumption. Therefore, condition (2) holds.

Finally, consider the inclusions

$$(A_{i+1} \cap B) \subseteq A_{i+1} \subseteq A_i \text{ and } (A_{i+1} \cap B) \subseteq A_{i+1} \subseteq H.$$

Since $[A_i : A_i \cap B] \geq \aleph_0$, $[A_i : A_{i+1} \cap B] \geq \aleph_0$. Moreover, since $A_i \cap B \not\sim H$, $[H : A_{i+1} \cap B] \geq \aleph_0$. However, by Proposition 5.2, at least one of $[H : A_{i+1}]$ and $[A_i : A_{i+1}]$ is finite, as either $H \precsim A_i$ or $A_i \precsim H$. Therefore, from

$$[A_i : A_{i+1} \cap B] = [A_i : A_{i+1}][A_{i+1} : A_{i+1} \cap B] \geq \aleph_0$$
$$[H : A_{i+1} \cap B] = [H : A_{i+1}][A_{i+1} : A_{i+1} \cap B] \geq \aleph_0$$

we obtain $[A_{i+1} : A_{i+1} \cap B] \geq \aleph_0$. Hence, condition (3) holds. This completes the construction, showing that $T$ is not convexly orderable. ☐

Before turning to the classification of VC-minimal abelian groups, we will need two lemmas. Both address the question of transferring VC-minimality between an abelian group and its direct summands. For groups $\mathfrak{A} = (A; +)$ and $\mathfrak{B} = (B; +)$, let $\mathfrak{A} \oplus \mathfrak{B} = (A \oplus B; +)$.

**Lemma 5.12.** *If $\mathfrak{B}$ is any abelian group and* $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$ *is VC-minimal, then* $\mathrm{Th}(\mathfrak{A})$ *is VC-minimal.*

*Proof.* Assume $T^* = \mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$ is VC-minimal. By Theorem 5.11 (3), $T^*$ is dp-minimal and $A \oplus B$ is upwardly coherent. By the proof of Lemma 5.10, there exists $\mathcal{H} \subseteq \mathrm{PP}(A \oplus B)$ such that $(\mathcal{H}; \subseteq)$ is a linear order and, for all $k$ and $m$, $\varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B})$ is a finite union of cosets of some $H_{k,m} \in \mathcal{H}$. Thus, we may write

$$\varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B}) = \bigcup_{i \leq n} (a_i \oplus b_i) + H_{k,m}$$

for some choice of $a_i \in A$, $b_i \in B$.

If $\pi_A$ denotes the projection of $\mathfrak{A} \oplus \mathfrak{B}$ onto $\mathfrak{A}$, note that $\pi_A(\varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B})) = \varphi_{k,m}(\mathfrak{A})$. So, clearly, $\mathcal{H}_A = \pi_A(\mathcal{H})$ is also linearly ordered by $\subseteq$. Moreover, we have

$$\varphi_{k,m}(\mathfrak{A}) = \bigcup_{i \leq n} a_i + \pi_A(H_{k,m}).$$

Therefore, using $\mathcal{H}_A$ in Lemma 5.4, we see that $T = \mathrm{Th}(\mathfrak{A})$ is VC-minimal. □

**Lemma 5.13.** *If $\mathfrak{B}$ is a finite abelian group, then* $\mathrm{Th}(\mathfrak{A})$ *is VC-minimal if and only if* $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$ *is VC-minimal.*

*Proof.* Suppose $T = \mathrm{Th}(\mathfrak{A})$ is VC-minimal. Again recalling the proof of Lemma 5.10, there exists $\mathcal{H} \subseteq \mathrm{PP}(A)$ so that $(\mathcal{H}; \subseteq)$ is a chain and, for all $k$ and $m$, $\varphi_{k,m}(\mathfrak{A})$ is a finite union of cosets of some $H_{k,m} \in \mathcal{H}$. For each $H \in \mathcal{H}$, choose a subgroup $B(H) \subseteq B$ minimal (with respect to $\subseteq$) such that $H \oplus B(H) \in \mathrm{PP}(A \oplus B)$. Finally, let

$$\mathcal{H}^* = \{H \oplus B(H) \mid H \in \mathcal{H}\}.$$

We verify that $\mathcal{H}^*$ satisfies the hypotheses of Lemma 5.4 for $\mathfrak{A} \oplus \mathfrak{B}$.

First, to see that $\mathcal{H}^*$ is a linear order under $\subseteq$, suppose $H_1 \subseteq H_2$ from $\mathcal{H}$. As

$$(H_1 \oplus B(H_1)) \cap (H_2 \oplus B(H_2)) = H_1 \oplus (B(H_1) \cap B(H_2))$$

is again an element of $\mathrm{PP}(A \oplus B)$, the minimality of $B(H_1)$ implies $B(H_1) = B(H_1) \cap B(H_2)$. Thus $B(H_1) \subseteq B(H_2)$ and $H_1 \oplus B(H_1) \subseteq H_2 \oplus B(H_2)$.

Second, we wish to show that $\varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B})$ is a boolean combination of cosets of elements of $\mathcal{H}^*$. Since we already know that $\varphi_{k,m}(\mathfrak{A})$ is a finite union of cosets of $H_{k,m}$, and $B$ is finite, it suffices to show that

$$H_{k,m} \oplus B(H_{k,m}) \subseteq \varphi_{k,m}(\mathfrak{A}) \oplus \varphi_{k,m}(\mathfrak{B}) = \varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B}).$$

That is, we need to show $B(H_{k,m}) \subseteq \varphi_{k,m}(\mathfrak{B})$. If not, however,

$$(H_{k,m} \oplus B(H_{k,m})) \cap \varphi_{k,m}(\mathfrak{A} \oplus \mathfrak{B}) = H_{k,m} \oplus (B(H_{k,m}) \cap \varphi_{k,m}(\mathfrak{B}))$$

would be in $\mathrm{PP}(A \oplus B)$, in which case $B(H_{k,m}) \cap \varphi_{k,m}(\mathfrak{B})$ would contradict the minimality of $B(H_{k,m})$.

Therefore, $\mathcal{H}^*$ satisfies the conditions of Lemma 5.4, proving VC-minimality of $\mathrm{Th}(\mathfrak{A} \oplus \mathfrak{B})$. The converse follows immediately from Lemma 5.12. □

We are now ready to prove an analog to Proposition 5.3 for VC-minimal (and convexly orderable) theories of abelian groups. The proposition gives a strong starting point, a complete list of dp-minimal theories of abelian groups. Theorem 5.11 and the above lemmas provide a set of tools for determining which of these are VC-minimal.

**Theorem 5.14.** *$T$ is VC-minimal (and convexly orderable) if and only if $\mathfrak{A}$ is elementarily equivalent to one of the following abelian groups:*

*(1)* $\bigoplus_{p \text{ prime}} \left(\mathbb{Z}(p^\infty)^{(\beta_p)}\right) \oplus \left(\mathbb{Z}_{(q)}\right)^{(\gamma)} \oplus \mathbb{Q}^{(\delta)} \oplus B$ *for a fixed prime $q$, finite abelian group $B$, and cardinals $\beta_p$, $\gamma$, and $\delta$ such that $\beta_p < \aleph_0$ for all $p \neq q$;*

*(2)* $\bigoplus_{p \text{ prime}} \left(B_p \oplus \mathbb{Z}(p^\infty)^{(\beta_p)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)}\right) \oplus \mathbb{Q}^{(\delta)}$ *for a fixed prime $q$, finite $p$-groups $B_p$, and cardinals $\beta_p$, $\gamma_p$, and $\delta$ such that $\beta_p < \aleph_0$ for all $p \neq q$ and $\gamma_p < \aleph_0$ for all $p$ (incuding $q$);*

*(3)* $\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(\alpha)} \oplus \left(\mathbb{Z}/p^{k+1}\mathbb{Z}\right)^{(\beta)} \oplus B$ *for some prime $p$, $k \geq 1$, finite abelian group $B$, and cardinals $\alpha$ and $\beta$, at least one of which is infinite.*

*Proof.* Suppose $T$ is dp-minimal. By Proposition 5.3, it falls under one of two categories. $T$ is either the theory of a group as in (3) above; or, $\mathfrak{A}$ is elementarily equivalent to

$$(\star) \qquad \bigoplus_{i\geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_{p.i})} \oplus \mathbb{Z}(p^\infty)^{(\beta_p)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)} \oplus B$$

for a prime $p$, nonsingular abelian group $B$, and cardinals $\alpha_{p,i}$, $\beta_p$, and $\gamma_p$ with each $\alpha_{p,i}$ finite.

For the former category, it follows from Corollary 5.6 and Lemma 5.13 that the group in (3) is also VC-minimal.

For the latter, first recall that by results of Szmielew [14], any abelian group is elementarily equivalent to one of the form

$$\bigoplus_{p \text{ prime}} \left(\bigoplus_{i\geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_{p,i})} \oplus \mathbb{Z}(p^\infty)^{(\beta_p)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)}\right) \oplus \mathbb{Q}^{(\delta)}.$$

It is straightforward to verify that such a group is only nonsingular if each $\alpha_{p,i}$, $\beta_p$, $\gamma_p$, and $\{i \mid \alpha_{p,i} > 0\}$ is finite. For instance, for $B = \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)}$, we have $B/pB = (\mathbb{Z}/p\mathbb{Z})^{(\gamma_p)}$, which is finite iff $\gamma_p$ is. Hence, $(\star)$ becomes

$$(\dagger) \qquad \bigoplus_{p \text{ prime}} \left(\bigoplus_{i\geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_{p,i})} \oplus \mathbb{Z}(p^\infty)^{(\beta_p)} \oplus \left(\mathbb{Z}_{(p)}\right)^{(\gamma_p)}\right) \oplus \mathbb{Q}^{(\delta)}$$

with each $\alpha_{p,i}$ finite and $\beta_p$, $\gamma_p$, and $\{i \mid \alpha_{p,i} > 0\}$ finite for $p \neq q$. In other words, writing $B_p = \bigoplus_{i\geq 1} \left(\mathbb{Z}/p^i\mathbb{Z}\right)^{(\alpha_{p,i})}$, we have that $B_p$ is a finite $p$-group for all $p \neq q$.

Suppose, then, that $(\dagger)$ is VC-minimal. We show that $(\dagger)$ is as in (1) or (2). By Corollary 5.8 and Lemma 5.12, $B_q$ must also be finite. If $\gamma_q < \aleph_0$, then we are in case (2).

Thus, suppose that $\gamma_p \geq \aleph_0$. Notice that $qA \precsim\!\!\!\!\!\not\;\; A$. We must show that $B = \bigoplus_p B_p$ is finite and $\gamma_p = 0$ for $p \neq q$.

If $\gamma_p > 0$ for some $p \neq q$, then $qA \not\precsim p^n A$ for all $n$. However, there is no $H \in \mathrm{PP}(A)$ with $H \sim qA$ such that $H \subseteq p^n A$ for all $n$. Therefore, the $\sim$-class of $qA$ is not upwardly coherent, contradicting Theorem 5.11.

If $B_p$ is nonzero for infinitely many primes $p$, let $p_0, p_1, \ldots$ enumerate all such primes, excluding $q$. Then we have

$$qA \precnsim \left( \prod_{i \leq n} p_i \right) A.$$

But there is no $H \in \mathrm{PP}(A)$ with $H \sim qA$ such that $H \subseteq \left( \prod_{i \leq n} p_n \right) A$ for every $n$, again contradicting upward coherence of the $\sim$-class of $qA$.

We have thus established that the theory of a VC-minimal abelian group belongs to one of the cases (1), (2), or (3). It remains only to show that the groups in (1) and (2) are indeed VC-minimal.

For both cases, $A[q^n]$ witnesses the upward coherence of its $\sim$-class for every $n$. In case (2), $kA \sim A$ for all $k$, so the chain of $\widetilde{\mathrm{PP}}(A)$ is given by

$$0 \precnsim A[q] \precnsim A[q^2] \precnsim \ldots \precnsim A,$$

and each $\sim$-class is upwardly coherent. Furthermore, each group in $\mathrm{PP}(A)$ is a boolean combination of cosets of groups in this chain. The details of this computation can be found in Lemma 5.28 of [3].

In case (1), in addition to $A[q^n]$, we also have that $q^n A$ witnesses the upward coherence of its $\sim$-class. The chain of $\widetilde{\mathrm{PP}}(A)$ is given by

$$0 \precnsim A[q] \precnsim A[q^2] \precnsim \ldots \precnsim q^2 A \precnsim qA \precnsim A.$$

Again, we refer to Lemma 5.28 of [3] to see that the groups in this chain generate every member of $\mathrm{PP}(A)$.

In both cases, therefore, $A$ is upwardly coherent. By Theorem 5.11, $T$ is VC-minimal. □

**Acknowledgments.** We would like to express our thanks to the referee for pointing out an error in the original draft of Proposition 5.3 and consequently, Theorem 5.14.

## References

[1] U. Andrews, S. Cotter, J. Freitag, and A. Medvedev, *VC-minimality: Examples and Observations*, in preparation.

[2] H. Adler, *Theories controlled by formulas of Vapnik-Chervonenkis codimension 1*, Preprint (2008).

[3] M. Aschenbrenner, A. Dolich, D. Haskell, D. MacPherson, and S. Starchenko, *Vapnik-Chervonenkis density in some theories without the independence property, II*, Notre Dame J. Formal Logic (to appear).

[4] S. Cotter and S. Starchenko, *Forking in VC-minimal theories*, J. Symbolic Logic **77** (2012), no. 4, 1257–1271.

[5] M. A. Dickmann, *Elimination of quantifiers for ordered valuation rings*, J. Symbolic Logic **52** (1987), no. 1, 116–128.

[6] A. Dolich, J. Goodrick, and D. Lippel, *Dp-minimality: basic facts and examples*, Notre Dame J. Formal Logic **52** (2011), no. 3, 267–288.

[7] J. Flenner and V. Guingona, *Canonical forests in directed families*, Proc. Amer. Math. Soc. (to appear).

[8] V. Guingona and M. C. Laskowski, *On VC-Minimal Theories and Variants*, Arch. Math. Logic (to appear).

[9] J. E. Holly, *Canonical forms for definable subsets of algebraically closed and real closed valued fields*, J. Symbolic Logic **60** (1995), no. 3, 843–860.

[10] D. MacPherson, D. Marker, and C. Steinhorn, *Weakly o-minimal structures and real closed fields*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5435–5483.
[11] A. Pillay and C. Steinhorn, *Definable sets in ordered structures I*, Trans. Amer. Math. Soc. **295** (1986), no. 2, 565–592.
[12] M. Prest, *Model Theory and Modules*, London Mathematical Society Lecture Note Series, vol. 130, Cambridge University Press, Cambridge, 1988.
[13] P. Simon, *On dp-minimal ordered structures*, J. Symbolic Logic **76** (2011), 448–460.
[14] W. Szmielew, *Elementary properties of Abelian groups*, Fund. Math. **41** (1955), 203–271.

University of Saint Francis, 2701 Spring Street, Fort Wayne, IN 46808, U.S.A.
*E-mail address*: `jflenner@sf.edu`

University of Notre Dame, Department of Mathematics, 255 Hurley Hall, Notre Dame, IN 46556, U.S.A.
*E-mail address*: `guingona.1@nd.edu`
*URL*: `http://www.nd.edu/~vguingon/`