

ON A COMMON GENERALIZATION OF SHELAH'S 2-RANK, DP-RANK, AND O-MINIMAL DIMENSION.

VINCENT GUINGONA,
CAMERON DONNAY HILL

ABSTRACT. In this paper, we build a dimension theory related to Shelah's 2-rank, dp-rank, and o-minimal dimension. We call this dimension op-dimension. We exhibit the notion of the n -multi-order property, generalizing the order property, and use this to create op-rank, which generalizes 2-rank. From this we build op-dimension. We show that op-dimension bounds dp-rank, that op-dimension is sub-additive, and op-dimension generalizes o-minimal dimension in o-minimal theories.

INTRODUCTION

At the beginning of this century, the study of dependent/NIP theories experienced something of a renaissance after a number of years of dormancy. With the exception of o-minimal theories (which are, of course, dependent, but this fact saw little actual use), most model theorists' attention had been directed towards stable and then simple theories. However, many "natural" algebraic examples turn out to be unstable, non-o-minimal but dependent (sometimes with stronger conditions than bare NIP), such as p -adic fields, definably compact groups arising in o-minimal structures, and ordered abelian groups. S. Shelah initiated a careful study of dependent/NIP theories in the series of papers, [16–19]; in this work, he defined certain sub-classes of dependent theories known (aptly) as strongly-dependent theories. A key tool in the development of strong-dependence is notion of dp-rank, which to some degree, resembles the notion of weight in a stable theory.

dp-Rank, $\text{dpR}(-)$, though it really is not a dimension, has some aspects that make it dimension-like. In particular, dp-rank is sub-additive in the sense that $\text{dpR}(ab/C) \leq \text{dpR}(a/C) + \text{dpR}(b/C)$, but without some significant effort to understand forking-dependence for a type of finite dp-rank, it can be relatively difficult initially to see dpR as a geometric construct. It is also somewhat difficult to accommodate

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dpR in the universe of pre-existing model-theoretic definitions. For example, a stable theory need not be strongly-dependent (have finite dp-rank for all types). Thus, while strong-dependence and dp-rank inherit ideas and intuitions from stability theory, they do not formally generalize it. Finally, dp-rank does not (to our knowledge) fit into the evolving framework of generalized-indiscernible “collapse” characterizations of model-theoretic dividing lines. That is to say, stability is equivalent to collapsing indiscernible sequences (linear orders) to indiscernible sets; dependence is equivalent to collapsing ordered-graph indiscernibles to indiscernible sequences; but there is no obvious analog even for theories of bounded dp-rank.

In this article, we define an analog of dp-rank – op-dimension, opD – that seems to remedy some of these “deficiencies.” Building atop a family of local op-ranks, we find that op-dimension has a number of intuitively desirable properties, including the following:

- The local ranks opR_n ($0 < n < \omega$) naturally generalize Shelah’s 2-rank $R(-, -, 2)$ to “multi-orders” and “multi-cuts”; in fact, these opR_n ’s formally generalize the classical 2-rank in that $R(-, -, 2) = \text{opR}_1(-, -)$.
- op-Dimension has a loosely, but still explicitly topological flavor. Indeed, in an o-minimal theory, the op-dimension of a definable set is identical to its o-minimal dimension (which also equals its dp-rank), and any theory that “sub-interpretable” in an o-minimal theory (in certain weak sense) must be of finite op-dimension.

Moreover, opD retains the dimension-like aspects of dp-rank over all strongly-dependent theories; that is, opD has the appropriate monotonicity properties, and it is sub-additive.

- The condition of bounded op-dimension for a theory can be (fruitfully, it seems) understood as a generalization of stability to multi-orders. For each n , we will find ourselves with an n -multi-order property (n -MOP), and 1-MOP is precisely the classical order property.

We will also see that the op-dimension of a type can be characterized in a manner very similar to the definition of dp-rank – simply replacing ICT-patterns with the closely related IRD-patterns; from this observation, we will show that op-dimension is always bounded by dp-rank. Thus, it appears that op-dimension has a part to play in any strongly-dependent theory.

It should be noted that op-dimension closely resembles what Shelah calls “ $\kappa_{\text{ird}}(T)$ ” (see Definition III.7.1 of [15]). This is also discussed

in Section 5 of [1]. We will touch on this fact more when we discuss IRD-patterns.

It is beyond the scope of this paper, but we must also remark, that the condition of bounded op-dimension *can* be characterized by “collapse” of certain generalized indiscernibles (in the sense of [8]); a little more precisely, op-dimension n is equivalent to asserting that every indiscernible $(n + 1)$ -multi-order collapses to an indiscernible n -multi-order. (As this result is an example of a rather more general phenomenon, we will save it for a more extended discussion of the latter; see [6].)

0.1. Outline of the Article. In Section 1, we outline the basic definitions and results surrounding op-rank and op-dimension. We introduce various ways of viewing op-dimension, first as a generalization of 2-rank, then through the lens of the multi-order property, then through its relationship to dp-rank and IRD-patterns. In Section 2, we give two proofs of the sub-additivity of op-dimension. The first proof follows the path of [10] using a modified notion of mutually indiscernible sequences. The second proof uses the multi-order property and has the flavor of a stability argument (e.g., Lascar’s inequality). Finally, in Section 3, we look at op-dimension in the special case of o-minimal theories. We show that op-dimension, dp-rank, and o-minimal dimension coincide in this case and we discuss interpretations of structures with finite op-dimension in o-minimal structures.

0.2. Notation. In this paper, we will work in a language \mathcal{L} , a complete \mathcal{L} -theory T , and a monster model \mathbb{U} . We will denote tuples of variables by x (instead of \bar{x}). By \mathbb{U}^x we mean all elements of \mathbb{U} of the sort of x . For a formula $\varphi(x)$, we let $\varphi(x)^1 = \varphi(x)$ and $\varphi(x)^0 = \neg\varphi(x)$. For a set $A \subseteq \mathbb{U}$, let $e\text{diag}(A)$ denote the elementary diagram of A , which is a set of $\mathcal{L}(A)$ formulas. Let $\text{diag}(A)$ denote the atomic diagram of A . Let 1_S denote the identity permutation on a set S .

Given a formula $\varphi(x, y)$ and a set $B \subseteq \mathbb{U}^y$, let $S_\varphi(B)$ denote the set of all φ -types over B , by which we mean maximally consistent subsets of the set

$$\{\varphi(x, b)^t : t < 2, b \in B\}.$$

The *independence dimension* of φ is the size of the largest finite set $B \subseteq \mathbb{U}^y$ so that

$$|S_\varphi(B)| = 2^{|B|}.$$

If no such largest set exists, we say that φ has the *independence property* (IP). If it does exist, we say φ has *NIP* (sometimes called “dependent”). A theory has *NIP* if all formulas have NIP.

1. DEFINITIONS AND BASIC RESULTS

Either “under the hood” or explicitly, the notions of multi-order and multi-cut together play an important role in much of the work in this article. In part to motivate these definitions, we begin our discussion with a somewhat eccentric definition of classical order property (which by compactness, is equivalent to the usual statement). In practice, we will work with “ n -multi-orders,” but in particular, a linear order is a 1-multi-order. In a linear order $(B, <)$, of course, a *cut* is a subset $X \subseteq B$ such that for all $b_0, b_1 \in B$, if $b_0 < b_1$ and $b_1 \in X$, then $b_0 \in X$.

Definition 1.1. Let $\varphi(x, y)$ be some formula of $\mathcal{L}(\mathbb{U})$. We say that $\varphi(x, y)$ has the order property if there is an indiscernible sequence $(a_q)_{q \in \mathbb{Q}}$ (of sort x) such that for every cut Z in \mathbb{Q} , there is a $b_Z \in \mathbb{U}^y$ such that $Z = \{q \in \mathbb{Q} : \models \varphi(a_q, b_Z)\}$.

Somewhat strangely, an n -multi-order is not (in general) just the cartesian product of n linear orders. (Otherwise, we would not have invented the terminology, obviously.) Instead, an n -multi-order is a set B equipped with n linear orderings that do not *a priori* have any dependencies. Further, rather than working with arbitrary multi-orders, it is much more convenient to observe, firstly, that the common universal theory of all n -multi-orders, MLO_n (for some fixed $0 < n < \omega$) has a model-companion MLO_n^* that has enough in common with the theory of $(\mathbb{Q}, <)$ to be useful to us (in fact, $\text{MLO}_1^* = \text{DLO}$).

Definition 1.2. For each $0 < n < \omega$, we define two closely related theories MLO_n and MLO_n^* with signature $\{<_0, \dots, <_{n-1}\}$, where each $<_i$ is a binary relation symbol. MLO_n asserts that each $<_i$ is a linear order of the universe *and nothing else*.

It is not difficult to verify that the class K_n of all finite models of MLO_n is a Fraïssé class. We take $\mathcal{A}_n = (A, <_0^{\mathcal{A}_n}, \dots, <_{n-1}^{\mathcal{A}_n})$ to be the countably infinite generic model (or Fraïssé limit) associated with K_n , and we define $\text{MLO}_n^* = \text{Th}(\mathcal{A}_n)$. Then MLO_n^* is just the model-companion of MLO_n , and by old results (see [9]), MLO_n^* is \aleph_0 -categorical and eliminates quantifiers.

Fact 1.3. Consider the $\{<_i\}_{i < n}$ on \mathbb{Q}^n in which,

$$<_i^{\mathbb{Q}^n} = \{(\bar{a}, \bar{b}) \in \mathbb{Q}^n \times \mathbb{Q}^n : a_i < b_i\}$$

for each $i < n$. Then, \mathbb{Q}^n is not a model of MLO_n^* . To see this, one may note that (for example),

$$\text{MLO}_2^* \models \forall xy \left(x \neq y \rightarrow \bigwedge_{i < 2} (x <_i y \vee y <_i x) \right)$$

but in \mathbb{Q}^2 , $(0, 1) \neq (0, 2)$ but $(0, 1) \not\prec_0 (0, 2)$ and $(0, 2) \not\prec_0 (0, 1)$.

Definition 1.4. For $0 < n < \omega$, any model of MLO_n is called an *n-multi-order*.

Now, if $\mathcal{B} = (B, <_0, \dots, <_{n-1})$ is a model of MLO_n , then a multi-cut (an *n-multi-cut*) in \mathcal{B} is a tuple (X_0, \dots, X_{n-1}) such that X_i is a cut in the reduct $(B, <_i)$ for each $i < n$.

To conclude these introductory remarks, we note that the potential to define all cuts in an indiscernible copy of $(\mathbb{Q}, <)$ is captured by Shelah's 2-rank, and insofar as MLO_n^* is similar enough to DLO, much of the insight of this article lies in the observation that analogous ranks, opR_n , can be devised to capture the potential of defining all multi-cuts in a model of MLO_n^* .

1.1. op-Ranks and the op-Dimension of a Type. In this subsection, we introduce our analogs of Shelah's 2-rank – of which there will be one rank for each $0 < n < \omega$ corresponding to the number of independent linear orders in an *n-multi-order*. Several of the most basic facts about opR_n s are themselves totally analogous to those regarding the 2-rank with almost identical proofs. In our presentation, to begin with anyway, we recall the definitions associated with the 2-rank and remind the reader of the relevant facts, and then we give analogous definitions for opR_n and the corresponding facts (without proof as those demonstrations are almost identical).

Definition 1.5. For a (consistent) partial type $\pi(x)$ and a finite set Δ of partitioned formulas $\varphi(x, y) \in \mathcal{L}(\mathbb{U})$, we recall that the Shelah 2-rank of $\pi(x)$ with respect to Δ is defined as follows:

- $R(\pi, \Delta, 2) \geq 0$ in any case.
- For a limit ordinal λ , $R(\pi, \Delta, 2) \geq \lambda$ if $R(\pi, \Delta, 2) \geq \alpha$ for every $\alpha < \lambda$.
- For any ordinal α , $R(\pi, \Delta, 2) \geq \alpha + 1$ if there is an instance $\varphi(x, a)$ from Δ such that $R(\pi \cup \{\varphi(x, a)^t\}, \Delta, 2) \geq \alpha$ for both $t < 2$.

As usual, we define $R(\pi, \Delta, 2) = \infty$ to mean that $R(\pi, \Delta, 2) \geq \alpha$ for every ordinal α . When $\Delta = \{\varphi\}$ consists of a single formula, one usually writes $R(-, \varphi, 2)$ in place of $R(-, \{\varphi\}, 2)$.

For an ordinal β , $\Gamma_\beta(\pi, \varphi)$ is the following set of sentences (with new constant symbols $a_\sigma, b_{\sigma|\ell}$ for $\sigma \in 2^\beta$ and $\ell < \beta$):

$$\bigcup_{\sigma \in 2^\beta} \pi(a_\sigma) \cup \{\varphi(a_\sigma, b_{\sigma|\ell})^{\sigma(\ell)} : \sigma \in 2^\beta, \ell < \beta\}.$$

The first basic result about the 2-rank is the following (coming from straightforward applications of compactness and “coding tricks”).

Fact 1.6. Let $\pi(x)$ be a partial type, and let Δ be a finite set of formulas of $\mathcal{L}(\mathbb{U})$. Also, let $\varphi(x, y) \in \mathcal{L}(\mathbb{U})$.

- (1) By compactness, $R(\pi, \Delta, 2) = \infty$ if and only if $R(\pi, \Delta, 2) \geq \omega$.
- (2) For any ordinal β , $R(\pi, \varphi, 2) \geq \beta$ if and only if $\Gamma_\beta(\pi, \varphi) \cup \text{eddiag}(\mathbb{U})$ is consistent.

Also, for any finite set Δ of formulas $\theta(x, y)$ of $\mathcal{L}(\mathbb{U})$, there is a single formula $\varphi_\Delta(x, z)$ of $\mathcal{L}(\mathbb{U})$ such that $R(-, \Delta, 2) = R(-, \varphi_\Delta, 2)$.

Now, we turn to our family of analogs of the 2-rank. For each parameter $0 < n < \omega$, the “key” distinction between the 2-rank and opR_n lies in replacing the trees $2^{<\omega}$ – whose nodes are maps $\sigma : k \rightarrow 2$ ($k < \omega$) – with trees $(2^n)^{<\omega}$ whose nodes are of the form $\sigma : k \rightarrow 2^n$ ($k < \omega$); an element of 2^n , here, represents a particular multi-cut in a model of MLO_n .

Definition 1.7. For $0 < n < \omega$, a (consistent) partial type $\pi(x)$ and a finite set Δ of partitioned formulas $\varphi(x, y) \in \mathcal{L}(\mathbb{U})$, we define $\text{opR}_n(\pi, \Delta)$ as follows:

- $\text{opR}_n(\pi, \Delta) \geq 0$ in any case.
- For a limit ordinal λ , $\text{opR}_n(\pi, \Delta) \geq \lambda$ if $\text{opR}_n(\pi, \Delta) \geq \alpha$ for every $\alpha < \lambda$.
- For any ordinal α , $\text{opR}_n(\pi, \Delta) \geq \alpha + 1$ if there are instances

$$\varphi_0(x, a_0), \dots, \varphi_{n-1}(x, a_{n-1}) \text{ from } \Delta$$

such that for each $\sigma \in 2^n$,

$$\text{opR}_n \left(\pi(x) \cup \left\{ \bigwedge_{i < n} \varphi_i(x, a_i)^{\sigma(i)} \right\}, \Delta \right) \geq \alpha$$

Again, we define $\text{opR}_n(\pi, \Delta) = \infty$ to mean that $\text{opR}_n(\pi, \Delta) \geq \alpha$ for every ordinal α . When $\Delta = \{\varphi\}$ consists of a single formula, we write $\text{opR}_n(-, \varphi)$ in place of $\text{opR}_n(-, \{\varphi\})$.

For an ordinal β , $\Gamma_{n,\beta}(\pi, \varphi)$ is the following set of sentences (with new constant symbols $a_\sigma, b_{\sigma,\ell,0}, \dots, b_{\sigma,\ell,n-1}$ for $\sigma = (\sigma_\ell)_{\ell < \beta} \in (2^n)^\beta$ and $\ell < \beta$):

$$\bigcup_{\sigma \in (2^n)^\beta} \pi(a_\sigma) \cup \{ \varphi(a_\sigma, b_{\sigma,\ell,i})^{\sigma_\ell(i)} : \sigma \in (2^n)^\beta, \ell < \beta \}.$$

Implicitly, we require that for all $\sigma, \tau \in (2^n)^\omega$, $\ell < \omega$, if $\sigma_k = \tau_k$ for each $k < \ell$, then $b_{\sigma,\ell,i} = b_{\tau,\ell,i}$ for each $i < n$.

Fact 1.8. Let $\pi(x)$ be a partial type, and let Δ be a finite set of formulas of $\mathcal{L}(\mathbb{U})$. Also, let $\varphi(x, y) \in \mathcal{L}(\mathbb{U})$ and $0 < n < \omega$.

- (1) $\text{opR}_n(\pi, \Delta) = \infty$ if and only if $\text{opR}_n(\pi, \Delta) \geq \omega$.
- (2) For any ordinal β , $\text{opR}_n(\pi, \varphi) \geq \beta$ if and only if $\Gamma_\beta(\pi, \varphi) \cup \text{eddiag}(\mathbb{U})$ is consistent.

Also, for any finite set Δ of formulas $\theta(x, y)$ of $\mathcal{L}(\mathbb{U})$, there is a single formula $\varphi_\Delta(x, z)$ of $\mathcal{L}(\mathbb{U})$ such that $\text{opR}_n(-, \Delta) = \text{opR}_n(-, \varphi_\Delta)$.

The closest analog to op-dimension in the stability theory literature is the notion of $\kappa_{\text{ird}}(T)$ defined in [15]. However, this concept is approached through the notion of an IRD-pattern and not through a 2-rank-like construction. In the unstable setting, using op-ranks, we can define op-dimension in a very simpleminded way.

Definition 1.9. For a partial type $\pi(x)$, we define the *op-dimension* of $\pi(x)$ to be,

$$\text{opD}(\pi) = \sup \{0 < n < \omega : (\exists \Delta) \text{opR}_n(\pi, \Delta) = \infty\} \leq \omega.$$

(Note that, by definition of sup on ordinals, $\sup \emptyset = 0$.) As is standard, for $a \in \mathbb{U}$ and $B \subset \mathbb{U}$, we define $\text{opD}(a/B)$ to be $\text{opD}(\text{tp}(a/B))$. For a formula $\varphi(x) \in \mathcal{L}(\mathbb{U})$, we define $\text{opD}(\varphi) = \text{opD}(\{\varphi\})$, and if X is the subset of \mathbb{U}^x defined by $\varphi(x)$, then $\text{opD}(X) = \text{opD}(\varphi)$.

Remark 1.10. Let us say that a partial type $\pi(x)$ is *unstable* if there are a formula $\varphi(x, y)$ of $\mathcal{L}(\mathbb{U})$ and an indiscernible sequence $(a_q)_{q \in \mathbb{Q}}$ of realizations of π such that for every cut X of $(\mathbb{Q}, <)$, there is a $b \in \mathbb{U}^y$ such that $\{q \in \mathbb{Q} : \mathbb{U} \models \varphi(a_q, b)\} = X$. Obviously, we should say that $\pi(x)$ is stable just in case it is not unstable. Thus, $\pi(x)$ is stable if and only if $\text{opD}(\pi) = 0$.

The following statement collects together a number of facts whose analogs for the 2-rank are essential in developing the machinery of forking-dependence in a stable theory – when one carries out that development using ranks, as turned out to be very useful for generalizations to simple and rosy theories. For our purposes, they immediately suggest that opD can indeed be viewed as a dimension function insofar as it has, at least, the appropriate monotonicity properties of a reasonable dimension theory.

Fact 1.11. opR_n ($0 < n < \omega$) has the following monotonicity properties:

- (1) Suppose $\pi_0(x) \subseteq \pi_1(x)$, $\Delta_0 \supseteq \Delta_1$, and $0 < n_0 \leq n_1 < \omega$. Then,

$$\text{opR}_{n_0}(\pi_0, \Delta_0) \geq \text{opR}_{n_1}(\pi_1, \Delta_1).$$

- (2) Let X_0, X_1 be definable sets of the same sort, $0 < n < \omega$, and Δ a finite set of formulas of $\mathcal{L}(\mathbb{U})$. Then
- $$\text{opR}_n(X_0 \vee X_1, \Delta) = \max \{ \text{opR}_n(X_0, \Delta), \text{opR}_n(X_1, \Delta) \}.$$
- (3) Let X, Y be a type-definable sets, and suppose $f : X \rightarrow Y$ is definable bijection. Then, for any $0 < n < \omega$, for any finite set Δ of $\mathcal{L}(\mathbb{U})$ -formulas, there is another finite set of formulas Δ' such that $\text{opR}_n(X, \Delta) = \text{opR}_n(Y, \Delta')$.

Corollary 1.12. *opD has the following monotonicity properties of a dimension (for type-definable sets X, Y):*

- (1) *If X, Y are in definable bijection with each other, then $\text{opD}(X) = \text{opD}(Y)$.*
- (2) *If $X \subseteq Y$, then $\text{opD}(X) \leq \text{opD}(Y)$.*
- (3) *Provided the definable sets X, Y are of the same sort,*

$$\text{opD}(X \vee Y) = \max \{ \text{opD}(X), \text{opD}(Y) \}.$$

1.2. Generalized Indiscernibles and n -MOP. In this subsection, we demonstrate some connections between op-dimension and an evolving framework connecting generalized-indiscernible “collapse” theorems and dividing lines in the model-theoretic (in)stability hierarchy.

Theorem 1.13. *For every $0 < n < \omega$, MLO_n is a theory of generalized indiscernibles in the sense of [6, 8, 11]:*

Let $\mathcal{A} \models \text{MLO}_n^$, and let \mathcal{M} be some $|A|^+$ -saturated \mathcal{L} -structure (in any language \mathcal{L} whatever). Let EM be a map $A^{<\omega} \rightarrow M^{<\omega}$ (really, a family of maps $A^k \rightarrow M^k$ for $0 < k < \omega$) such that:*

- *If $EM(a_0, \dots, a_{k-1}) = (b_0, \dots, b_{k-1})$, then for each $\sigma \in \text{Sym}(k)$,*

$$EM(a_{\sigma(0)}, \dots, a_{\sigma(k-1)}) = (b_{\sigma(0)}, \dots, b_{\sigma(k-1)}).$$

- *For any $\bar{a}, \bar{a}' \in A^{<\omega}$, $\text{tp}^{\mathcal{M}}(EM(\bar{a} \widehat{\ } \bar{a}')) = \text{tp}^{\mathcal{M}}(EM(\bar{a}) \widehat{\ } EM(\bar{a}'))$.*

Then, there is a map $g : A \rightarrow M$ such that:

- *For all $0 < k < \omega$ and $\bar{a}, \bar{a}' \in A^k$,*

$$\text{qftp}^A(\bar{a}) = \text{qftp}^A(\bar{a}') \Rightarrow \text{tp}^{\mathcal{M}}(g\bar{a}) = \text{tp}^{\mathcal{M}}(g\bar{a}').$$

- *For all $0 < k < \omega$, every $\bar{a} \in A^k$, and every finite set Δ of \mathcal{L} -formulas, there is an $\bar{a}' \in A^k$ such that $\text{qftp}^A(\bar{a}) = \text{qftp}^A(\bar{a}')$ and $\text{tp}_{\Delta}^{\mathcal{M}}(g\bar{a}) = \text{tp}_{\Delta}^{\mathcal{M}}(EM(\bar{a}'))$.*

(For brevity, we say that g is an indiscernible picture of \mathcal{A} in \mathcal{M} patterned on EM .)

The proof of Theorem 1.13 can be found in [8]. For background on the generalized-indiscernible collapse phenomenon, we cite the following theorem of [14].

Theorem 1.14. *Let OG be the theory (in the signature $\{\prec^{(2)}, R^{(2)}\}$) of ordered graphs; that is, OG asserts the following:*

- “ \prec is a linear order of the universe (i.e. of the vertices).”
- $\forall xy [R(x, y) \rightarrow ((x \neq y) \wedge R(y, x))]$

Then, OG has a model-companion OG^ , which is also the theory of the Fraïssé limit of the class of all finite ordered graphs. Moreover:*

- (1) *OG is a theory of indiscernibles in the same sense (of Theorem 1.13) that each MLO_n is.*
- (2) *The following are equivalent for any complete theory T in any language whatever:*
 - (a) *T is dependent/NIP.*
 - (b) *For any indiscernible picture g of a model $\mathcal{A} = (A, \prec^A, R^A)$ of OG^* in an model \mathcal{M} of T , $(g(a))_{a \in A}$ is an indiscernible sequence in order type (A, \prec^A) , in the usual sense.*

Intuitively, this theorem asserts that, for all intents, the theory of indiscernibles OG encodes the independence property. Viewing the theory of linear order LO ($= \text{MLO}_1$) as a theory of indiscernibles, as we may, the following venerable characterization of stability also fits (loosely) into this framework. (There is actually a mismatch in that the “remainder” of OG in a dependent/NIP theory is MLO_1 , which is still a theory of indiscernibles, but the remainder of MLO_1 in a stable theory is the theory of equality, which, in fact, is not a theory of indiscernibles.)

Theorem 1.15. *Let T be a complete theory in any language. The following are equivalent:*

- (1) *T is unstable.*
- (2) *In some model of T , there is an indiscernible sequence $(a_q)_{q \in \mathbb{Q}}$ that is not an indiscernible set.*

We now define the (“smoothed”) combinatorial property that seems to correspond to our op-dimensions in the same way that the order property corresponds to 2-rank.

Definition 1.16. Let $0 < n < \omega$, and let $\pi(x)$ be a consistent partial type. We say that $\pi(x)$ has the *n -multi-order property (n -MOP)* if there are an indiscernible picture (\mathcal{A}_n, g) in $\pi(\mathbb{U})$ and a formula $\varphi(x, y)$ of $\mathcal{L}(\mathbb{U})$ such that for any multi-cut (X_0, \dots, X_{n-1}) of \mathcal{A}_n , there are $b_0, \dots, b_{n-1} \in \mathbb{U}^y$ such that $X_i = \{a \in A : \mathbb{U} \models \varphi(g(a), b_i)\}$ for each $i < n$.

We note that the “collapse” results in the previous two theorems require a rather fine analysis of exactly how, for example, an ordered-graph indiscernible picture can collapse down to an indiscernible picture of reduct. Such an analysis for our n -MOPs would take us outside of the scope of the goals of this paper, though such an analysis will be given in [6]. For now, we consider a more basic analog of the following fact:

Fact 1.17. A partial type $\pi(x)$ is stable iff $R(\pi, \varphi, 2) < \omega$ for every formula $\varphi(x, y)$ of $\mathcal{L}(\mathbb{U})$ iff $\text{opR}_1(\pi, \varphi) < \omega$ for every formula $\varphi(x, y)$ iff $\text{opD}(\pi) = 0$.

Proposition 1.18. *Let $0 < n < \omega$, and let $\pi(x)$ be a consistent partial type. Then, $\text{opD}(\pi) \geq n$ if and only if $\pi(x)$ has n -MOP.*

Proof. (“only if”) Assuming $\text{opD}(\pi(x)) \geq n$, let $\varphi(x, y)$ be some formula of $\mathcal{L}(\mathbb{U})$ such that $\text{opR}_n(\pi(x), \varphi) = \infty$. Thus, $\text{eddiag}(\mathbb{U}) \cup \Gamma_{n, \omega}(\pi, \varphi)$ is consistent; we recover two families

$$\{a_\sigma : \sigma \in (2^n)^\omega\}, \{b_{\sigma, \ell, i} : \sigma \in (2^n)^\omega, \ell < \omega, i < n\}$$

such that:

- Each a_σ is a realization of $\pi(x)$.
- For any $\sigma, \tau \in (2^n)^\omega$, $\ell < \omega$, and $i < n$, if $\sigma_j = \tau_j$ for each $j < \ell$, then $b_{\sigma, \ell, i} = b_{\tau, \ell, i}$.
- For any $\sigma \in (2^n)^\omega$, $\ell < \omega$, and $i < n$, $\models \varphi(a_\sigma, b_{\sigma, \ell, i})^{\sigma_\ell(i)}$

Now, we observe that if $B = (A, <_0, \dots, <_{n-1})$ is a finite model of MLO_n with, say, $|B| = N < \omega$, then there is an *embedding* $A \rightarrow (N^n, <_0, \dots, <_{n-1})$, where in the latter structure, the orders are interpreted coordinate-wise. By Theorem 1.13 (and the fact that \mathbb{U} is \aleph_1 -saturated), we obtain an injective mapping $g : A \rightarrow \mathbb{U}$ such that $g[A] \subseteq \pi(\mathbb{U})$ and for every multi-cut (X_0, \dots, X_{n-1}) of \mathcal{A}_n , there are $b_0, \dots, b_{n-1} \in \mathbb{U}^y$ such that $X_i = \{a \in A : \models \varphi(g(a), b_i)\}$ for each $i < n$. Thus, $\pi(x)$ has n -MOP.

(“if”) Suppose $\pi(x)$ has n -MOP, and let $g : A \rightarrow \mathbb{U}$ and $\varphi(x, y)$ witness this fact. We will show that $\text{opR}_n(\pi, \varphi) = \infty$, and for this, it is enough to show that for each $N < \omega$, $\Gamma_{n, N}(\pi, \varphi) \cup \text{eddiag}(\mathbb{U})$ is consistent. We observe that for any $N < \omega$, there is an *injective homomorphism* of the coordinate-wise ordered structure $((2^N)^n, <_0, \dots, <_{n-1})$ into \mathcal{A}_n , and this suffices for the consistency of $\Gamma_{n, N}(\pi, \varphi) \cup \text{eddiag}(\mathbb{U})$, as required. \square

1.2.1. *A Remark on Localized opD.* We now remark briefly on a localization of op-dimension to finite sets of formulas. It will probably come as no surprise that such a localized rendition of op-dimension amounts to little more than a restatement of the independence property.

Definition 1.19. Let $\pi(x)$ be a partial type. For a finite set Δ of $\mathcal{L}(\mathbb{U})$ formulas, we define,

$$\text{opD}(\pi, \Delta) = \sup \{0 < n < \omega : \text{opR}_n(\pi, \Delta) = \infty\} \leq \omega.$$

Proposition 1.20. *The theory $T = \text{Th}(\mathbb{U})$ has the independence property if and only if $\text{opD}(\{x=x\}, \Delta) = \omega$ for some tuple x and some finite set Δ of formulas of $\mathcal{L}(\mathbb{U})$.*

Proof. Assuming $\varphi(x, y)$ has the independence property in T , we show that $\text{opD}(x=x, \varphi) = \omega$. We may grant ourselves an indiscernible sequence $(e_a : a \in A)$ (where A is the universe of \mathcal{A}_n equipped with the first order $<_0^{\mathcal{A}_n}$) such that for every $Z \subseteq A$, there is some $b_Z \in \mathbb{U}^y$ such that $\{a : \models \varphi(e_a, b_Z)\} = Z$. Given $0 < n < \omega$, let $g : A \rightarrow \mathbb{U}$ be an indiscernible picture of \mathcal{A}_n in \mathbb{U} patterned on

$$\text{EM} : A^{<\omega} \rightarrow \mathbb{U}^{<\omega} : (a_0, \dots, a_{k-1}) \mapsto (e_{a_0}, \dots, e_{a_{k-1}}).$$

Then, again, for every $Z \subseteq A$, there is a $b_Z \in \mathbb{U}^y$ such that $\{a : \models \varphi(g(a), b_Z)\} = Z$. Since (of course) multi-cuts are subsets of A , this demonstrates that $x=x$ has n -MOP via $\varphi(x, y)$, so $\text{opD}(\{x=x\}, \Delta) \geq n$

Conversely, suppose $\text{opD}(\{x=x\}, \Delta) = \omega$ for some tuple x and some finite set Δ of formulas of $\mathcal{L}(\mathbb{U})$. Without loss of generality, we may assume that $\text{opD}(\{x=x\}, \varphi) = \omega$ for some single formula $\varphi(x, y)$. For a set B of size $N < \omega$, there are $N!$ linear orders on B . Enumerating all of these orders $<_0^B, \dots, <_{N!-1}^B$, we find ourselves with a finite substructure of $\mathcal{A}_{N!}$. Thus, for any $d < \omega$ one can find arbitrarily large finite sets $B \subset \mathbb{U}^x$ such that

$$|S_\varphi(B)| = 2^{|B|}$$

showing that the independence dimension of $\varphi(x, y)$ is unbounded – i.e. $\varphi(x, y)$ has the independence property. \square

1.3. op-Dimension as an Analog of dp-Rank: ICT- and IRD-patterns. Thus far, we have seen op-dimension through the lens of the “stability-like” analysis of op-ranks and n -MOP. On the other hand, op-dimension can also be characterized using analysis similar to that done on dp-rank; indeed, op-dimension n can be seen as a close analog of dp-rank n . With this in mind, we introduce another alternative definition of op-dimension. Compare this to the definition of dp-rank given by Definition 2.1 and 2.2 of [5].

Theorem 1.21. *Fix a partial type $\pi(x)$ over a parameter set A and $n < \omega$. The following are equivalent:*

- (1) $\text{opD}(\pi) \leq n$;

- (2) For all formulas $\varphi(x, y)$, for all indiscernible sequences $\langle b_q : q \in \mathbb{Q} \rangle$ over A , and all $a \models \pi$, there exists $C_0 < \dots < C_n$ a convex partition of \mathbb{Q} such that, for each $i \leq n$, the set

$$\{q \in C_i : \models \varphi(a, b_q)\}$$

is either finite or cofinite in C_i .

Proof. (1) \Rightarrow (2): Suppose that $\text{opD}(\pi) > n$, hence π has $(n + 1)$ -MOP. Fix an indiscernible picture (\mathcal{A}_{n+1}, g) in $\pi(\mathbb{U})$ and a formula $\varphi(x, y)$ witnessing this. For each $i \leq n$, let \mathcal{C}_i be the set of all $<_i$ -cuts of A , and let $\mathcal{C} = \prod_{i \leq n} \mathcal{C}_i$. We can multi-order \mathcal{C} via

$$\langle X_0, \dots, X_n \rangle \leq_i \langle X'_0, \dots, X'_n \rangle \text{ iff } X_i \subseteq X'_i.$$

Moreover, for each $c = \langle X_0, \dots, X_n \rangle \in \mathcal{C}$, choose $b_c = \langle b_{c,0}, \dots, b_{c,n} \rangle \in \mathbb{U}_y^{n+1}$ such that, for all $i \leq n$,

$$X_i = \{a \in A : \models \varphi(g(a), b_{c,i})\}.$$

Finally, choose a sequence c_0, c_1, \dots from \mathcal{C} such that, if $j < k < \omega$, then $c_j <_i c_k$ for all $i \leq n$. Let $f : (n + 1) \rightarrow \omega$ be any function. By $<_i$ -density of A for each $i \leq n$, there exists $a \in A$ such that

$$\models \varphi(g(a), b_{c_j,i}) \text{ iff } j > f(i)$$

for each $i \leq n$. By compactness and Ramsey's Theorem, there exists $\langle b'_q : q \in \mathbb{Q} \rangle$ indiscernible and $a' \models \pi$ such that

$$\models \varphi(a', b'_{q,i}) \text{ iff } q > i$$

for each $i \leq n$. Let $\psi(x; y_0, \dots, y_n)$ be the formula that holds if evenly many of $\varphi(x, y_i)$ holds for $i \leq n$. Let \sim be the natural convex equivalence relation on \mathbb{Q} generated by ψ , namely

$$q \sim r \text{ iff } (\forall q')(q < q' \leq r \Rightarrow \models [\psi(a', b'_q) \leftrightarrow \psi(a', b'_{q'})]).$$

Then, \sim has exactly $n + 1$ classes, each infinite. Thus, we see that $\langle b'_q : q \in \mathbb{Q} \rangle$, a , and $\psi(x, \bar{y})$ is a witness to the failure of (2).

(2) \Rightarrow (1): Suppose (2) fails, witnessed by $\varphi(x, y)$, $\langle b_q : q \in \mathbb{Q} \rangle$, and $a \models \pi$. Since φ has NIP, we know it has finite alternation rank. Therefore, by possibly trimming down the sequence and replacing φ with $\neg\varphi$, we may assume that $C_0 < \dots < C_{n+1}$ is a convex partition of \mathbb{Q} , with each C_i infinite, and

$$\models \varphi(a, b_q) \text{ iff } q \in C_i \text{ for some } i \leq n + 1 \text{ even.}$$

Now we show that $\text{opR}_{n+1}(\pi, \varphi) = \infty$, showing that $\text{opD}(\pi) > n$.

Fix $K < \omega$ and choose $\bar{\sigma} = \langle \sigma_k : k < K \rangle \in ({}^{n+1}2)^K$. Suppose we have constructed $q_{i,k} \in (C_i \cup C_{i+1})$ for each $i \leq n$ and $k < K$ so that

$q_{i,k} \in C_i$ if and only if $\sigma_k(i) = 0$. Moreover, assume that if $k_0 < k_1 < K$, then

- (1) $\sigma_{k_0}(i) = \sigma_{k_1}(i) = 0$ implies $q_{i,k_0} < q_{i,k_1}$, and
- (2) $\sigma_{k_0}(i) = \sigma_{k_1}(i) = 1$ implies $q_{i,k_1} < q_{i,k_0}$.

That is, the sequences $q_{i,k}$ approach the cut between C_i and C_{i+1} . For each $i \leq n$, choose $q_i^* \in \mathbb{Q}$ anywhere between all the $q_{i,k}$ for $\sigma_k(i) = 0$ and the $q_{i,k}$ for $\sigma_k(i) = 1$. Thus, by indiscernibility,

$$\pi(x) \cup \{\varphi(x, b_{q_i, k})^{i + \sigma_k(i) \pmod{2}} : k < K, i \leq n\} \cup \{\varphi(x, b_{q_i^*})^{\eta(i)} : i \leq n\}$$

is consistent for all $\eta \in {}^{n+1}2$. Hence, by induction,

$$\text{opR}_{n+1}(\pi, \varphi) \geq K.$$

Since K was arbitrary, we see that $\text{opR}_{n+1}(\pi, \varphi) = \infty$, as desired. \square

We define the notion of an IRD-pattern, given in Definition III.7.1 of [15] and Section 5 of [1], which closely resembles an ICT-pattern (used for dp-rank). In [15], Shelah notes that ‘‘IRD’’ is an abbreviation for ‘‘independent orders.’’ Shelah only considers infinite IRD-patters, but we will diverge from this and consider only finite patterns.

Definition 1.22. Fix a partial type $\pi(x)$, $n < \omega$, and α an ordinal. Consider a sequence of formulas $\bar{\psi} = \langle \psi_i(x, y_i) : i < n \rangle$ and a sequence $\bar{b} = \langle b_{j,i} : j < \alpha, i < n \rangle$ where each $b_{j,i}$ is of the same sort as y_i . We say that $\langle \bar{\psi}, \bar{b} \rangle$ forms an *IRD-pattern in $\pi(x)$ of depth n and length α* if, for all $f : n \rightarrow \alpha$, the following type is consistent

$$\pi(x) \cup \{\neg \psi_i(x, b_{j,i}) : i < n, j < f(i)\} \cup \{\psi_i(x, b_{j,i}) : i < n, f(i) \leq j < \alpha\}.$$

Lemma 1.23. *Fix a partial type $\pi(x)$. Then $\text{opD}(\pi) \leq n$ if and only if there exists no IRD-pattern in π of depth $n + 1$ and length ω .*

Proof. (\Rightarrow): Suppose there exists an IRD-pattern in π of depth $n + 1$ and length ω , say $\langle \bar{\psi}, \bar{b} \rangle$. Let $\varphi(x; y_0, \dots, y_n)$ be the formula that holds if and only if an even number of $\psi_i(x, y_i)$ hold and let $c_j = \langle b_{j,0}, \dots, b_{j,n} \rangle$ for each $j < \omega$. For each strictly monotonic $f : (n + 1) \rightarrow \omega$, the type

$$(1) \quad \pi(x) \cup \{\varphi(x, c_j) : f(i - 1) \leq j < f(i) \text{ for even } i \leq n + 1\} \cup \{\neg \varphi(x, c_j) : f(i - 1) \leq j < f(i) \text{ for odd } i \leq n + 1\},$$

is consistent, where we interpret $f(-1) = 0$ and $f(n + 1) = \omega$. By Ramsey’s Theorem and compactness, we may assume that the c_q are indexed by $q \in \mathbb{Q}$ and that $\bar{c} = \langle c_q : q \in \mathbb{Q} \rangle$ is indiscernible. Fix $f : (n + 1) \rightarrow \mathbb{Q}$ such that $f(i) = i$ for all $i \leq n$ and fix a a realization of (1). Then \bar{c} and a are witnesses to the fact that $\pi(x)$ has op-dimension $> n$ (as in Theorem 1.21 (2)).

(\Leftarrow): Suppose $\pi(x)$ has op-dimension $> n$, witnessed by $\varphi(x, y)$, $\langle b_q : q \in \mathbb{Q} \rangle$, and $a \models \pi$ (as in Theorem 1.21 (2)). Since φ is NIP, φ has finite alternation rank, hence there exists a minimal finite convex partition \mathcal{C} of \mathbb{Q} so that, for each $C \in \mathcal{C}$, there exists $D \subseteq C$ cofinite in C such that, for all $q, r \in D$, $\models \varphi(a, b_q) \leftrightarrow \varphi(a, b_r)$. Since this is a witness to the op-dimension being greater than n , there exists $C_0 < C_1 < \dots < C_{n+1}$ from \mathcal{C} with alternating majority truth value of $\varphi(a, b_q)$. Let $\psi_i(x, y)$ be either $\varphi(x, y)$ or $\neg\varphi(x, y)$ such that, for all $i \leq n$ and cofinitely many $q \in C_i$, $\models \psi_i(a, b_q)$ if and only if i is odd. By indiscernibility over A and compactness, we see that $\langle \psi_i : i \leq n \rangle$ together with $\langle b_{i+1/(j+1)} : j < \omega, i \leq n \rangle$ form an IRD-pattern of depth $n + 1$ and length ω in $\pi(x)$. \square

We see now that there is an obvious relationship between dp-rank and op-dimension.

Definition 1.24. Fix a partial type $\pi(x)$, $n < \omega$, and α an ordinal. Consider a sequence of formulas $\bar{\psi} = \langle \psi_i(x, y_i) : i < n \rangle$ and a sequence $\bar{b} = \langle b_{j,i} : j < \alpha, i < n \rangle$ where each $b_{j,i}$ is of the same sort as y_i . We say that $\langle \bar{\psi}, \bar{b} \rangle$ forms an *ICT-pattern in $\pi(x)$ of depth n and length α* if, for all $f : n \rightarrow \alpha$, the following type is consistent

$$\pi(x) \cup \{ \neg\psi_i(x, b_{j,i}) : i < n, j < \alpha, j \neq f(i) \} \cup \{ \psi_i(x, b_{f(i),i}) : i < n \}.$$

We say that a type $\pi(x)$ has *dp-rank $\geq n$* if there exists an ICT-pattern in π of depth n and length ω . We denote this by $\text{dpR}(\pi) \geq n$.

The next proposition is straightforward, and implicitly shown in [1], but we give a proof here for completeness.

Proposition 1.25. *Let $\pi(x)$ be a partial type with finite dp-rank. Then,*

$$\text{opD}(\pi) \leq \text{dpR}(\pi).$$

Proof. Fix $n > \omega$ and let $\bar{\psi} = \langle \psi_i(x, y_i) : i < n \rangle$ together with $\bar{b} = \langle b_{j,i} : j < \omega, i < n \rangle$ be an IRD-pattern of depth n and length ω in $\pi(x)$. Let

$$\varphi_i(x; y_{0,i}, y_{1,i}) = \neg[\psi_i(x, y_{0,i}) \leftrightarrow \psi_i(x, y_{1,i})]$$

and let

$$c_{j,i} = \langle b_{2j,i}, b_{2j+1,i} \rangle.$$

Notice that $\langle \varphi_i : i < n \rangle$ together with $\langle c_{j,i} : j < \omega, i < n \rangle$ is an ICT-pattern of depth n and length ω in $\pi(x)$. Therefore, $\text{opD}(\pi) \geq n$ implies $\text{dpR}(\pi) \geq n$. \square

In particular, if T is dp-minimal (e.g., o-minimal), then $\text{opD}(\mathbb{U}) \leq \text{dpR}(\mathbb{U}) \leq 1$.

Many proofs in the literature establishing the existence of an ICT-pattern implicitly go through an IRD-pattern. For example, the proof Fact 2.7 of [4] first builds an IRD-pattern, then an ICT-pattern from it as in the proof of Proposition 1.25 above.

In [10], it is shown that dp-rank is sub-additive in the following sense:

$$\text{dpR}(\text{tp}(a, b/C)) \leq \text{dpR}(\text{tp}(a/C)) + \text{dpR}(\text{tp}(b/C \cup \{a\})).$$

This is proved using the technology of mutually indiscernible sequences. We adapt this for the op-dimension setting using something called almost mutually indiscernible sequences.

1.4. Almost-indiscernible Sequences.

Definition 1.26. Fix a set X , a collection of sequences

$$\mathcal{J} = \{\langle b_{j,i} : j \in J_i \rangle : i \in X\},$$

and a set of formulas

$$\Delta(y_{k,i})_{k < K_i, i \in X}.$$

We say that \mathcal{J} is Δ -mutually-indiscernible if, for all sequences

$$j_{0,i} < \dots < j_{K_i-1,i} \text{ and } \ell_{0,i} < \dots < \ell_{K_i-1,i}$$

from J_i for each $i \in X$, and for all $\delta \in \Delta$, we have that

$$\models \delta(b_{j_{k,i},i})_{k < K_i, i \in X} \leftrightarrow \delta(b_{\ell_{k,i},i})_{k < K_i, i \in X}.$$

(We note that this depends heavily on the partition of variables in formulas in Δ .) For a set of parameters A , we say that \mathcal{J} is *almost mutually indiscernible over A* if, for each formula δ over A as above, there exists $J'_i \subseteq J_i$ finite for each $i \in X$ such that the collection of sequences

$$\{\langle b_{j,i} : j \in (J_i \setminus J'_i) \rangle : i \in X\}$$

is δ -mutually-indiscernible. We say that $\langle b_j : j \in J \rangle$ is *almost indiscernible over A* if $\{\langle b_j : j \in J \rangle\}$ is almost mutually indiscernible over A (where $|X| = 1$).

Lemma 1.27. *Let $\mathcal{J} = \{\langle b_{j,i} : j \in J_i \rangle : i \in X\}$ be a set of almost mutually indiscernible sequences, $\delta(y_{k,i})_{k < K_i, i \in X}$ any formula (over any parameter set), and $\sigma_i : \omega \rightarrow J_i$ a strictly monotone function for each $i \in X$. Then, there exists $M_i < \omega$ for each $i \in X$ and $t < 2$ such that, for all $M_i < j_{0,t} < \dots < j_{K_i-1,t} < \omega$ for each $i \in X$,*

$$\models \delta(b_{\sigma_i(j_{k,t},i)})_{k < K_i, i \in X}^t.$$

That is, there is a “limit truth value” for δ under $\langle \sigma_i : i \in X \rangle$.

Proof. Write δ as $\delta(a; y_{k,i})_{k < K_i, i \in X}$ for $\delta(x; y_{k,i})_{k < K_i, i \in X}$ a formula over \emptyset . Since T is NIP, δ is NIP, so suppose it has independence dimension $< N$.

Suppose the conclusion fails. We build a sequence with alternating truth values on δ to get a contradiction. First, choose for each $i \in X$,

$$0 < j_{0,i}^0 < \dots < j_{K_i-1,i}^0 < \omega$$

arbitrarily so that $\models \neg\delta(a; b_{\sigma_i(j_{k,i}^0),i})_{k < K_i, i \in X}$ (by assumption, this exists). Now, suppose that $j_{0,i}^\ell < \dots < j_{K_i-1,i}^\ell$ is constructed for $\ell \geq 0$, $i \in X$ so that

$$\models \delta(a; b_{\sigma_i(j_{k,i}^\ell),i})_{k < K_i, i \in X}^{\ell \pmod{2}}.$$

Let $M_i = j_{K_i-1,i}^\ell$. By assumption, these $\langle M_i : i \in X \rangle$ and $t = \ell \pmod{2}$ do not satisfy the conclusion. Therefore, there exists, for each $i \in X$,

$$M_i < j_{0,i}^{\ell+1} < \dots < j_{K_i-1,i}^{\ell+1} < \omega$$

such that

$$\models \delta(a; b_{\sigma_i(j_{k,i}^{\ell+1}),i})_{k < K_i, i \in X}^{\ell+1 \pmod{2}}.$$

Notice that, for each $\eta \in {}^N 2$, the formula

$$\theta_\eta(z_0, \dots, z_{N-1}) = \exists x \left(\bigwedge_{w < N} \delta(x; z_w)^{\eta(w)} \right)$$

is over \emptyset . Therefore, by almost mutual indiscernibility of \mathcal{J} , we may assume that the sequence

$$\langle \langle b_{\sigma_i(j_{k,i}^\ell),i} : k < K_i, i \in X \rangle : \ell < \omega \rangle$$

is $\{\theta_\eta : \eta \in {}^N 2\}$ -indiscernible (since \mathcal{J} is merely *almost* mutually indiscernible, we may have to remove a finite portion of the beginning). Now, for each $\eta \in {}^N 2$, we have by definition

$$\models \bigwedge_{\ell < N} \delta(a; b_{\sigma_i(j_{k,i}^{2\ell+\eta(\ell)}),i})_{k < K_i, i \in X}^{\eta(\ell)}.$$

By $\{\theta_\eta : \eta \in {}^N 2\}$ -indiscernibility, we get that

$$\models \exists x \left(\bigwedge_{\ell < N} \delta(x; b_{\sigma_i(j_{k,i}^\ell),i})_{k < K_i, i \in X}^{\eta(\ell)} \right).$$

Since η was arbitrary, this contradicts the fact that δ has independence dimension $< N$. \square

In particular, if $J_i = \omega$ for all $i \in X$ and \mathcal{J} is almost mutually indiscernible over \emptyset , then \mathcal{J} is almost mutually indiscernible over any set of parameters. We use this to develop the notion of limit types.

Definition 1.28. Let $\mathcal{J} = \{\langle b_{j,i} : j \in J_i \rangle : i \in X\}$ be an almost mutually indiscernible sequence over a parameter set A , $\bar{y} = \langle y_{k,i} : k < K_i, i \in X \rangle$ a tuple of variables, and $\sigma_i : \omega \rightarrow J_i$ a strictly monotone function for each $i \in X$. Then, for any set of parameters B , define the *limit type of \mathcal{J} in the variables \bar{y} under $\bar{\sigma} = \langle \sigma_i : i \in X \rangle$* as follows: For $\delta(\bar{y})$ over B ,

$$\delta(\bar{y}) \in \lim_{\bar{\sigma}}(\mathcal{I}/B)(\bar{y})$$

if and only if there exists $M_i < \omega$ for each $i \in X$ such that, for all $M_i < j_{0,i} < \dots < j_{K_i-1,i} < \omega$ for each $i \in X$,

$$\models \delta(b_{\sigma_i(j_{k,i}),i})_{k < K_i, i \in X}.$$

By Lemma 1.27 above, this is a complete type over B in the variables \bar{y} (that is consistent by compactness).

Fix an ordinal $\alpha < \omega^2$ and let $M < \omega$ be maximal such that $\omega \cdot M \leq \alpha$. For each $m < M$, define the injection $\sigma_m : \omega \rightarrow \alpha$ as follows:

$$\sigma_m(i) = (\omega \cdot m) + i.$$

With this setup, we get the following lemma:

Lemma 1.29. *Fix a finite set X and fix a parameter set B . Suppose that $\mathcal{J} = \{\langle b_{j,i} : j \in \alpha \rangle : i \in X\}$ is almost mutually indiscernible over \emptyset but not almost mutually indiscernible over B . Then there exists $i_0 \in X$, $m_i < M$ for each $i \in X \setminus \{i_0\}$, $m_0^* < m_1^* < M$, and $\delta(\bar{y})$ over B such that*

- (1) $-\delta(\bar{y}) \in \lim_{\langle \sigma_{m_i : i \neq i_0} \rangle + \langle \sigma_{m_0^*} \rangle}(\mathcal{I}/B)(\bar{y})$ and
- (2) $\delta(\bar{y}) \in \lim_{\langle \sigma_{m_i : i \neq i_0} \rangle + \langle \sigma_{m_1^*} \rangle}(\mathcal{I}/B)(\bar{y})$.

Proof. Let $\delta(\bar{y})$ over B witness that \mathcal{J} is not almost mutually indiscernible over B . For each choice of $\bar{m} = \langle m_i : i \in X \rangle \in M^X$, consider the sequence of injections $\bar{\sigma}_{\bar{m}} = \langle \sigma_{m_i} : i \in X \rangle$. By Lemma 1.27, there exists $t_{\bar{m}} < 2$ such that

$$\delta^{t_{\bar{m}}}(\bar{y}) \in \lim_{\bar{\sigma}_{\bar{m}}}(\mathcal{I}/B)(\bar{y}).$$

If all values of $t_{\bar{m}}$ are equal, then, by removing finitely many elements, $\delta(\bar{y})$ has a constant value on \mathcal{I} . This contradicts the fact that δ witnesses that \mathcal{I} is not almost mutually indiscernible over B .

Therefore, there must exist \bar{m} and \bar{m}' such that $t_{\bar{m}} \neq t_{\bar{m}'}$. By switching one coordinate at a time, there exists $i_0 \in X$, \bar{m} , and \bar{m}' such that

- (1) $m_i = m'_i$ for all $i \neq i_0$,
- (2) $m_{i_0} < m'_{i_0}$, and
- (3) $t_{\bar{m}} \neq t_{\bar{m}'}$.

By possibly swapping δ for $-\delta$, we get the desired conclusion. \square

We use this lemma in the next section to derive the sub-additivity of opD .

2. SUB-ADDITIVITY OF opD

2.1. Using Almost Mutually Indiscernible Sequences. In this subsection, we use the machinery of almost mutually indiscernible sequences discussed above to show that opD -dimension is sub-additive. First, we prove a result analogous to Proposition 4.4 of [10].

Proposition 2.1. *For π a partial type over A , the following are equivalent*

- (1) $\text{opD}(\pi) \leq n$;
- (2) *For all $a \models \pi$, ordinals $\alpha < \omega^2$, $L < \omega$, and $\{\langle b_{j,i} : j \in \alpha \rangle : i < L\}$ almost mutually indiscernible over A , there exists $I \subseteq L$ with $|I| \geq L - n$ so that $\{\langle b_{j,i} : j \in \alpha \rangle : i \in I\}$ is almost mutually indiscernible over $A \cup \{a\}$.*

Proof. (1) \Rightarrow (2): We show the contrapositive, so suppose (2) fails, witnessed by $a \models \pi$ and $\{\langle b_{j,i} : j \in \alpha \rangle : i < L\}$ almost mutually indiscernible over A . Clearly $L > n$. Fix $N < \omega$ and $\theta(x) \in \pi(x)$ arbitrary.

Fix $d < L$ and let $X = \{d, \dots, L - 1\}$. So long as $|X| \geq L - n$ (i.e., $d \leq n$), by assumption, $\{\langle b_{j,i} : j \in \alpha \rangle : i \in X\}$ is not almost mutually indiscernible over $A \cup \{a\}$. By Lemma 1.29, there exists a formula $\delta_d(x; y_{k,i})_{k < K, i \in X}$ over A , $\ell < \omega$, $i_0 \in X$, $m_i < \omega$ for each $i \in X \setminus \{i_0\}$, and $m_0^* < m_1^* < \omega$ such that, for all

$$((\omega \cdot m_i) + \ell) < j_{0,i} < \dots < j_{K-1,i} < (\omega \cdot (m_i + 1)) \text{ for } i \in X \setminus \{i_0\},$$

all $t < 2$, and all $((\omega \cdot m_t^*) + \ell) < j_{0,i_0} < \dots < j_{K-1,i_0} < (\omega \cdot (m_t^* + 1))$, we have that

$$\models \delta_d(a; b_{j_{k,i},i})_{k < K, i \in X}^t.$$

Without loss of generality (rearranging the sequences), we may assume $i_0 = d$. If $d > 0$ and the σ_{d-1} have been constructed, then choose $\ell < \omega$ large enough so that no instance of $\sigma_{d-1}(k, i)$ lies in the intervals between $((\omega \cdot m_i) + \ell)$ and $(\omega \cdot (m_i + 1))$.

Define a function $\sigma_d : (2NK \times X) \rightarrow \alpha$. For each $d < i < L$ and $k < 2NK$, let

$$\sigma_d(k, i) = (\omega \cdot m_i) + \ell + 1 + k.$$

For $i = d$ and $k < NK$, let

$$\sigma_d(k, d) = (\omega \cdot m_0^*) + \ell + 1 + k.$$

For $i = d$ and $NK \leq k < 2NK$, let

$$\sigma_d(k, d) = (\omega \cdot m_1^*) + \ell + 1 + (k - NK).$$

Notice that $\sigma_d(k, i)$ is strictly increasing in the variable k . For $j < 2N$, define

$$c_{d,j} = \langle b_{\sigma_d(k+Kj,i),i} : k < K, i \in X \rangle.$$

This construction terminates when $d = n + 1$. We claim that δ_d together with $\langle c_{d,j} : j < N \rangle$ for $d \leq n$ form an IRD-pattern of depth $n + 1$ and length N in θ .

By construction, for each $d \leq n$, for all $j < 2N$, we have that

$$\models \delta_d(a; c_{d,j}) \text{ iff } j \geq N.$$

By almost mutual indiscernibility over A (and choosing our ℓ above sufficiently large), we get, for each $\eta : (n + 1) \rightarrow N$,

$$\models \exists x \left(\theta(x) \wedge \bigwedge_{d \leq n, j < N} \delta_d(x; c_{d,j})^{\text{iff } \eta(d) > j} \right).$$

This yields the desired conclusion.

Since N and θ were arbitrary, by compactness, $\text{opD}(p) \geq n + 1$.

(2) \Rightarrow (1): Suppose that $\text{opD}(p) \geq n + 1$, witnessed by an IRD-pattern $\bar{\psi} = \langle \psi_i(x, y_i) : i \leq n \rangle$ together with $\bar{b} = \langle b_{j,i} : j < \omega, i \leq n \rangle$. Let $\alpha = \omega \cdot 2$ and let \mathcal{L}' be the language \mathcal{L} expanded by constants $b_{j,i}$ for $j < \alpha$ and $i \leq n$ and a constant a . Let Σ be the \mathcal{L}' -theory expanding T which states that

- (i) $a \models \pi$,
- (ii) $\{ \langle b_{j,i} : j < \alpha \rangle : i \leq n \}$ is mutually indiscernible over A , and
- (iii) $\models \psi_i(a, b_{j,i})$ if and only if $\omega \leq j < \alpha$.

Any finite subset of Σ is realized (using Ramsey's Theorem for (ii)). Therefore, this is consistent. Finally, we show that, for all $i \leq n$, $\langle b_{j,i} : j \in \alpha \rangle$ is not almost indiscernible over $A \cup \{a\}$, witnessed by $\psi_i(a, y)$. By (iii), $\models \psi_i(a, b_{j,i})$ if and only if $\omega \leq j < \omega \cdot 2$. Therefore, for no finite $J_0 \subseteq \alpha$ do we have that $\langle b_{j,i} : j \in (\alpha \setminus J_0) \rangle$ is $\psi_i(a, y)$ -indiscernible. \square

Theorem 2.2 (Sub-additivity of op-dimension). *Suppose a and b are tuples and A is a set of parameters. Then,*

$$\text{opD}(a, b/A) \leq \text{opD}(a/A) + \text{opD}(b/A \cup \{a\}).$$

Proof. We use Proposition 2.1 in both directions. First, suppose that

$$\text{opD}(a/A) = n \text{ and } \text{opD}(b/A \cup \{a\}) = k,$$

let $\alpha < \omega^2$, and let $\mathcal{J} = \{\langle b_{j,i} : j \in \alpha \rangle : i < L\}$ be almost mutually indiscernible over A . Then, by Proposition 2.1, there exists $I \subseteq L$ with $|I| = L - n$ so that $\{\langle b_{j,i} : j \in \alpha \rangle : i \in I\}$ is almost mutually indiscernible over $A \cup \{a\}$. Now, by Proposition 2.1 again, there exists $I' \subseteq I$ with $|I'| = L - n - k$ such that $\{\langle b_{j,i} : j \in \alpha \rangle : i \in I'\}$ is almost mutually indiscernible over $A \cup \{a, b\}$. Since \mathcal{J} was arbitrary, by Proposition 2.1, this implies that $\text{opD}(a, b/A) \leq n + k$. \square

2.2. Alternative Proof of Sub-additivity Using MOPs. In this subsection, we present a sketch of an alternative proof of the fact that opD is sub-additive.¹ This proof uses our n -multi-order properties for the analysis of opD -dimension, and really amounts to one main compactness argument.

Second proof (sketch) of Theorem 2.2. Fix $0 < n < \omega$, and suppose $\text{opD}(e_0 e_1) \geq n$, where e_0, e_1 are elements of sorts v_0, v_1 in \mathbb{U} , and on the other hand, suppose $\text{opD}(e_0) = k_0 < n$. Of course, we must show that $\text{opD}(e_1/e_0) \geq k_1 = n - k_0$. Suppose $g = (g_0, g_1) : \mathcal{A}_n \rightarrow \mathbb{U}^{v_0} \times \mathbb{U}^{v_1}$ is an indiscernible picture of \mathcal{A}_n and $\varphi(v_0 v_1, u)$ is some formula of \mathcal{L} such that: (1) $g_0(a)g_1(a) \equiv e_0 e_1$ for all $a \in A$; and (2) for every n -multi-cut $Z = (Z_0, \dots, Z_{n-1})$, there are $c_0, \dots, c_{n-1} \in \mathbb{U}^u$ such that $Z_i = \{a \in A : \models \varphi(g_0(a), g_1(a), c_i)\}$ for each $i < n$.

For the compactness argument, we introduce a language \mathcal{L}^+ that accommodates several indiscernible pictures and many new constant symbols. (For brevity, we assume that \mathbb{U} has only one sort, and we (rather blithely) work with function symbols whose arities may be of dimension greater than one.)

- \mathcal{L}^+ has four sorts X_0, X_1, Y and M for $\mathcal{A}_{k_0}, \mathcal{A}_{k_1}, \mathcal{A}_n$, and \mathbb{U} , respectively. (In particular, these sorts have symbols for all of the necessary structure coming from those models.)
- \mathcal{L}^+ has function symbols

$$\mathbf{g}_i = Y \rightarrow M^{v_i}, \mathbf{f}_i : X_i \rightarrow Y, \mathbf{h}_i : X_i \rightarrow M^{v_i} \ (i < 2).$$

For economy, we may combine \mathbf{g}_0 and \mathbf{g}_1 into a single function symbol $\mathbf{g} : Y \rightarrow M^{v_0} \times M^{v_1}$.

- For each $i < 2$, \mathcal{L}^+ has constant symbols c_a^i of sort X_i for each $a \in A$.
- For each $i < 2$, each $\mathcal{B} \in \text{age}(\mathcal{A}_{k_i})$, and each k_i -multi-cut $Z = (Z_0, \dots, Z_{k_i-1})$ of \mathcal{B} , \mathcal{L}^+ has constant symbols $d_0^i(\mathcal{B}, Z), \dots, d_{k_i-1}^i(\mathcal{B}, Z)$.

¹Since the result is already proven, it seems unnecessary to subject the reader to another argument in full detail.

- \mathcal{L}^+ has two additional constant symbols e_0^*, e_1^* of sorts M^{v_0}, M^{v_1} , respectively.

Now, we will define a set Γ of \mathcal{L}^+ -sentences so that a model of Γ contains indiscernible pictures witnessing that $\text{tp}(a_1/a_0)$ has k_1 -MOP.

- (1) For $i < 2$, Γ asserts $X_i \models \text{MLO}_{k_i}$ (not $\text{MLO}_{k_i}^*$), and for each $\theta(a_0, \dots, a_{m-1}) \in \text{diag}(\mathcal{A}_{k_i})$, $\theta(c_{a_0}^i, \dots, c_{a_{m-1}}^i)$ is in Γ .
- (2) Γ asserts that (M, Y, \mathbf{g}) is elementarily equivalent to $(\mathbb{U}, \mathcal{A}_n, g)$.
- (3) For each $i < 2$, the sentence $\bigvee_{s:k_i \xrightarrow{1:1} n} \text{Emb}_i(s)$ is in Γ , where in turn, for each one-to-one map $s : k_i \rightarrow n$, $\text{Emb}_i(s)$ is the sentence asserting that $\mathbf{f}_i : X_i \rightarrow Y$ is an embedding up to identifying the order relations via $<_j \mapsto <_{s(j)}$.
- (4) For $i < 2$, for each formula $\psi(w_0, \dots, w_{m-1})$ of \mathcal{L} if $i = 0$ (of $\mathcal{L}(e_0^*)$ if $i = 1$), and for all elements a_0, \dots, a_{m-1} and b_0, \dots, b_{m-1} of A , if

$$\text{qftp}^{\mathcal{A}_{k_i}}(a_0, \dots, a_{m-1}) = \text{qftp}^{\mathcal{A}_{k_i}}(b_0, \dots, b_{m-1})$$

then the sentence $I_{\psi, a, b}^i =$

$$\psi(\mathbf{f}_i(c_{a_0}^i), \dots, \mathbf{f}_i(c_{a_{m-1}}^i)) \leftrightarrow \psi(\mathbf{f}_i(c_{b_0}^i), \dots, \mathbf{f}_i(c_{b_{m-1}}^i))$$

is in Γ .

- (5) $\psi(e_0^*) \in \Gamma$ for each $\psi(v_0) \in \text{tp}(e_0)$. Further, $\mathbf{h}_0 = \mathbf{g}_0 \circ \mathbf{f}_0$ and $\mathbf{h}_1 = \mathbf{g}_1 \circ \mathbf{f}_1$.
- (6) For $\mathcal{B} \in \text{age}(\mathcal{A}_{k_1})$ and $Z = (Z_0, \dots, Z_{k_1-1})$ a k_0 -multi-cut in \mathcal{B} , the following sentence $\varepsilon^1[\mathcal{B}, Z]$ is in Γ :

$$\bigvee \left\{ \begin{array}{l} \bigwedge \{ \varphi(\mathbf{g}_0(\mathbf{f}_1(c_b^1)), \mathbf{h}_1(c_b^1), d_j^1(\mathcal{B}, Z))^{b \in Z_j} : b \in B, j < k_1 \} \\ \bigwedge \{ \varphi(e_0^*, \mathbf{h}_1(c_b^1), d_j^1(\mathcal{B}, Z))^{b \in Z_j} : b \in B, j < k_1 \}. \end{array} \right.$$

We observe that if \mathcal{M} is a model of Γ , and \mathcal{A}_{k_1} is identified with its representation as the set of constant symbols $\{c_a^1 : a \in A\}$, then by the Pigeonhole Principle, one of two things can be true, either one of which demonstrates that $\text{opD}(e_1/e_0) \geq k_1$:

- (1) $\mathbf{h}_1^{\mathcal{M}} : \mathcal{A}_{k_1} \rightarrow M^{v_1}$ is an indiscernible picture of \mathcal{A}_{k_1} that with $\varphi'(v_1, u) = \varphi(e_0^*, v_1, u)$, shows that if $a \in A$, then

$$\text{opD}(\mathbf{h}_1^{\mathcal{M}}(a)/e_0^*) \geq k_1.$$

- (2) $\mathbf{h}_1^{\mathcal{M}} : \mathcal{A}_{k_1} \rightarrow M^{v_1}$ is an indiscernible picture of \mathcal{A}_{k_1} that with $\varphi''(v_1, uv_0) = \varphi(v_0, v_1, u)$, shows that if $a \in A$, then

$$\text{opD}(\mathbf{h}_1^{\mathcal{M}}(a)/e_0^*) \geq k_1.$$

In either case, $\text{opD}(e_1/e_0) \geq k_1$ follows because $e_0^* \mathbf{h}_i^M(a) \equiv e_0 e_1$. Thus, it is enough to verify that Γ is finitely satisfiable.

If Γ is *not* satisfiable, then there are $\mathcal{B}_i \in \text{age}(\mathcal{A}_{k_i})$ – say that $B_i = \{b_0^i <_0 \cdots <_0 b_{N-1}^i\}$ – formulas $\xi(v_0) \in \text{tp}(e_0)$, $\psi_0(w_0, \dots, w_{N-1}) \in \mathcal{L}$, and $\psi_1(w_0, \dots, w_{N-1}) \in \mathcal{L}(e_0^*)$, and a sentence σ of $\text{Th}(\mathbb{U}, \mathcal{A}_n, g)$ such that (up to abusing notation in the transfers $a \mapsto c_a^i$) for all one-to-one $s_i : k_i \rightarrow n$ and $t_i < 2$ ($i < 2$),

$$\begin{aligned} \sigma, \text{diag}(\mathcal{B}_0), \text{diag}(\mathcal{B}_1), \mathbf{h}_0 = \mathbf{g}_0 \circ \mathbf{f}_0, \mathbf{h}_1 = \mathbf{g}_1 \circ \mathbf{f}_1, \\ \text{Emb}_0(s_0), \text{Emb}_1(s_1), \psi_0(\mathbf{h}_0 B_0)^{t_0}, \psi_1(\mathbf{h}_1 B_1)^{t_1} \end{aligned}$$

implies

$$\neg \bigwedge \{ \varepsilon^1[\mathcal{B}_1, Z] : Z \text{ a } k_1\text{-multi-cut of } \mathcal{B}_1 \}.$$

A few moments' reflection will convince the reader that this contradicts the assumption that for every n -multi-cut $W = (W_0, \dots, W_{n-1})$ of \mathcal{A}_n , there are $c_0, \dots, c_{n-1} \in \mathbb{U}^u$ such that $W_i = \{a \in A : \models \varphi(g_0(a), g_1(a), c_i)\}$ for each $i < n$. This completes the proof sketch. \square

3. CONNECTIONS TO o-MINIMALITY

3.1. Equivalence of opD, dpR, and o-Minimal Dimension. The goal of this subsection is to show that op-dimension, dp-rank, and o-minimal dimension coincide in o-minimal theories. For a definable set X , the op-dimension of X is simply the op-dimension of the partial type $x \in X$, and this is denoted $\text{opD}(X)$. Similarly define the dp-rank.

Theorem 3.1. *If T is o-minimal (where $<$ is dense) and X is a definable set, then the op-dimension of X , the dp-rank of X , and the o-minimal dimension of X are equal.*

Proof. Suppose that $X \subseteq \mathbb{U}^m$ has o-minimal dimension $\geq n$. Then, there exists a projection $\pi : \mathbb{U}^m \rightarrow \mathbb{U}^n$ so that $\pi(X)$ has non-empty interior. That is, there exists an open box $B \subseteq \pi(X)$. Since the ordering $<$ is dense, there exists an embedding $\sigma : \omega^n \rightarrow B$. This extends to an embedding $\sigma' : \omega^n \rightarrow X$ via π^{-1} . Consider, for each $i < n$, the formula $\psi_i(x, y)$ that holds of $\langle a, b \rangle \in (\mathbb{U}^m)^2$ if and only if the i th coordinate of $\pi(a)$ is less than the i th coordinate of $\pi(b)$. Then, ψ_i together with $\langle \sigma(0, \dots, 0, j, 0, \dots, 0) : 0 < j < \omega \rangle$ (j in the i th coordinate) form an IRD-pattern of depth n in $x \in X$. Therefore, the op-dimension of X is $\geq n$. Moreover, by Proposition 1.25, $\text{dpR}(X) \geq \text{opD}(X) \geq n$.

Conversely, suppose the o-minimal dimension of $X \subseteq \mathbb{U}^m$ is $< n$. By Corollary 1.12 (3), we may suppose X is a cell. Then, there exists a definable injection $f : X \rightarrow \mathbb{U}^k$ for some $k < n$. Hence, by Corollary 1.12

(1) and (2), $\text{opD}(X) \leq \text{opD}(\mathbb{U}^k)$. Since we are working in an o-minimal theory, the op-dimension of \mathbb{U}^1 is ≤ 1 . Therefore, by Theorem 2.2, $\text{opD}(\mathbb{U}^k) \leq k < n$, hence $\text{opD}(X) < n$. Moreover, by sub-additivity of dp-rank (Theorem 4.8 of [10]), $\text{dpR}(X) \leq \text{dpR}(\mathbb{U}^k) \leq k < n$. \square

This result generalizes to any theory expanding dense linear order with a good cell decomposition. In fact, an interesting question is how does op-dimension relate to cell decomposition? Can one develop a notion of cell decomposition from the assumption that a theory expanding dense linear order has op-dimension ≤ 1 ?

Remark 3.2. Notice that dp-rank and op-dimension coincide on any distal theory (see Definition 2.1 of [21]). To see this, consider the characterization of op-dimension given in Theorem 1.21 together with the characterization of distality given in Lemma 2.7 of [21] (so called external characterization). From here one can see that global “point discrepancies” cannot exist. Since o-minimal theories are distal, this (along with the fact that dp-rank and o-minimal dimension coincide) gives another proof of Theorem 3.1.

3.2. d -Sub-interpretations in o-Minimal Structures. In this subsection, re-consider the dimension equivalence just presented in the language of interpretations between structures. Unsurprisingly, it turns out that a “true” interpretation of some \mathcal{B} in another structure \mathcal{M} is not quite appropriate, and instead we work with a mapping of B to onto a dense subset of a member of \mathcal{M}^{eq} . With this adjustment, we find that if “ d -sub-interpretation” of \mathcal{B} in the quotient of an n -dimensional definable set of \mathcal{M} exists, then $\text{opD}(\text{Th}(\mathcal{B})) \leq n - 1$ in essence, this is just a restatement of the results of the previous subsection. As a partial converse, however, we manage to show that every countable op-minimal theory T (in a one-sorted language) – meaning that $\text{opD}(T) \leq 1$ – is d -sub-interpretable (in fact, d -sub-definable) in 1-dimension in a pseudo-o-minimal theory.

Definition 3.3. Assume $\mathcal{M} = (M, <, \dots)$ is o-minimal. For some $0 < n < \omega$, let $D \subseteq M^n$ be a definable set with interior (with respect to the product o-minimal topology), and let $E \subseteq D \times D$ be a definable equivalence relation on D with quotient mapping $\pi_E : D \rightarrow D/E$. Then we shall always understand D/E to be endowed with the final topology induced by π_E ; that is, $U \subseteq D/E$ is open if and only if $\pi_E^{-1}U = \{d \in D : \pi_E(d) \in U\}$ is open in the subspace topology on D .

We now formulate our weakened notion of interpretability of a structure \mathcal{B} in a topological structure \mathcal{M} .

Definition 3.4. Fix a model \mathcal{B} of T , and let $\mathcal{M} = (M, <, \dots)$ be some o-minimal structure. The data of an n -dimensional d -sub-interpretation I of \mathcal{B} in \mathcal{M} is the following:

$$I = \left(X, E, (\varphi^I(v_0, \dots, v_{k-1}))_{\varphi(x_0, \dots, x_{k-1}) \in \text{QF}(\mathcal{L})}, f \right)$$

where $X \subseteq M^r$ ($n \leq r < \omega$) is definable of o-minimal dimension n ; $E \subseteq X \times X$ is a definable equivalence relation on X ; for each quantifier-free formula $\varphi(x_0, \dots, x_{k-1})$ of \mathcal{L} (i.e. $\varphi \in \text{QF}(\mathcal{L})$), $\varphi^I(v_0, \dots, v_{k-1})$ is a formula of $\mathcal{L}_{\mathcal{M}}$ such that $|v_i| = r$ for each $i < k$ and $\varphi^I(\mathcal{M}) \subseteq X^k$; and $f : B \rightarrow X/E$ is a one-to-one mapping. For these data to amount to an n -dimensional d -sub-interpretation, we require that:

- For each quantifier-free formula $\varphi(x_0, \dots, x_{k-1})$ of \mathcal{L} (i.e. each $\varphi \in \text{QF}(\mathcal{L})$), for all $b_0, \dots, b_{k-1} \in B$,

$$\mathcal{B} \models \varphi(b) \Leftrightarrow \mathcal{M} \models \exists v_0 \dots v_{k-1} \left(\bigwedge_{i < k} v_i \in f(b_i) \wedge \varphi^I(v_0, \dots, v_{k-1}) \right).$$

- $f[A]$ is dense in X/E .

Naturally enough, we will say that T is n -dimensionally o-minimally d -sub-interpretable if there are $\mathcal{B} \models T$, \mathcal{M} an o-minimal structure, and an n -dimensional d -sub-interpretation of \mathcal{B} in \mathcal{M} . When the equivalence relation E is trivial (i.e. $E = 1_X$), then we say “sub-definable” instead of “sub-interpretable.”

We remark that there is nothing exceedingly special about o-minimality in this definition (or the previous one). Indeed, largely the same formulations would work for weakly o-minimal, pseudo-o-minimal, or (it seems) any theory with a definable topology.

Fact 3.5. Let $\mathcal{B} \models T$, and let $I = (X, E, (\varphi^I), f)$ be an n -dimensional d -sub-interpretation of \mathcal{B} in an o-minimal structure \mathcal{M} . If $f[A] = X/E$, then I is an interpretation of \mathcal{A} in \mathcal{M} in the classical sense.

Theorem 3.6. *Assume T eliminates quantifiers (in a language with a single sort). If T is n -dimensionally o-minimally d -sub-interpretable, then $\text{opD}(T) \leq n$ – meaning that $\text{opD}(\{x=x\}) \leq n$ where x .*

Proof. Let $\mathcal{B} \models T$, and let $I = (X, E, (\varphi^I), f)$ be a d -sub-interpretation of \mathcal{B} in an o-minimal structure $\mathcal{M} = (M, <, \dots)$. Absorbing the parameters of the formulas X, E, φ^I into the language, we assume that I is over \emptyset . Also, assuming $\dim(X) \leq n-1$, we show that T cannot have n -MOP. For a contradiction, suppose (as we may, by QE) $\psi(x, y) \in \mathcal{L}$ is a quantifier-free formula and $g : A \rightarrow B$ is an indiscernible picture of \mathcal{A}_n in \mathcal{B} such that for every multi-cut (X_0, \dots, X_{n-1}) , there are b_i

($i < n$) such that $X_i = \{a : \mathcal{B} \models \psi(g(a), b_i)\}$. The following claim is a relatively straightforward consequence (by compactness) of the fact that $f[B]$ is dense in X/E .

Claim. *There are elementary extensions $\mathcal{A}_n \preceq \mathcal{A}'$, $\mathcal{B} \preceq \mathcal{B}'$, $\mathcal{M} \preceq \mathcal{M}'$ and functions $g' : A; \rightarrow B'$, and $f' : B \rightarrow X(\mathcal{M}')/E(\mathcal{M}')$ such that:*

- (1) $f' \subseteq f$, and $I = (X(\mathcal{M}'), E(\mathcal{M}'), (\varphi^I), f')$ is a d -sub-interpretation of \mathcal{B}' in \mathcal{M}' .
- (2) $g \subseteq g'$, and g' is an indiscernible picture of \mathcal{A}' in \mathcal{B}' patterned on $EM : \bar{a} \mapsto g\bar{a}$, and for every multi-cut (X_0, \dots, X_{n-1}) , there are $b_0, \dots, b_{n-1} \in B'$ such that $X_i = \{a : \mathcal{B}' \models \psi(g(a), b_i)\}$ for each $i < n$.
- (3) Relative to the 0-definable structure on $X(\mathcal{M}')/E(\mathcal{M}')$, the composition $(f' \circ g') \upharpoonright A$ is an indiscernible picture of \mathcal{A}_n in \mathcal{M}' .

Before sketching a demonstration the claim, we first complete the proof the theorem from it. (For clarity, we will abuse notation now by suggesting that x, y, w are really single variables rather than tuples; this a fiction due abbreviating.) As ψ is quantifier-free,

$$\mathcal{B}' \models \psi(b, b') \Leftrightarrow \mathcal{M}' \models \exists v, v' (v \in f'(b) \wedge v' \in f'(b') \wedge \psi^I(v, v')).$$

whenever $b, b' \in B$ are of the appropriate sorts. In particular, if (X_0, \dots, X_{n-1}) is a multi-cut in \mathcal{A}_n , then choosing $b_0, \dots, b_{n-1} \in B'$ appropriately, we have

$$\begin{aligned} X_i &= \{a \in A : \mathcal{B}' \models \psi(g'(a), b_i)\} \\ &= \{a \in A : \mathcal{M}' \models \exists v v' (v \in f'(g'(a)) \wedge v' \in f'(b_i) \wedge \psi^I(v, v'))\} \end{aligned}$$

Thus, the indiscernible picture $(f' \circ g') \upharpoonright A$ and the formula implicit above show that

$$n \leq \text{opD}(X) \leq \dim(X(\mathcal{M}')) = n - 1$$

which is impossible in light of Theorem 3.1.

Proof (sketch) of claim. We will work in a language with three sorts B, M, A on which the symbols of \mathcal{L} , $\mathcal{L}_{\mathcal{M}}$, and those of MLO_n , respectively, are imposed; between these sorts, we will also have function symbols $\mathbf{f} : A \rightarrow B$ and $\mathbf{g} : B \rightarrow X \subseteq M^r$. We include constants for all elements of the countable model \mathcal{A}_n . Finally, to account for defined multi-cuts, we include function symbols $\mathbf{h}_0, \dots, \mathbf{h}_{n-1} : A^n \rightarrow B$. Now, the truth of the claim boils down to verifying that the following set of sentences Γ of this language is finitely-satisfiable.

- Γ says B is a model of T , M is a model of $\text{Th}(\mathcal{M})$, and A is a model of MLO_n^* with the countable model as a substructure via the added constants.

- For each $k < \omega$ and each quantifier-free-complete k -type $q(x)$ of the language of MLO_n , for each formula $\varphi(x) \in \text{tp}^{\mathcal{B}}(ga)$ where $a \in q(\mathcal{A}_n)$,

$$(\forall x_0 \dots x_{k-1} \in A) (q(x) \rightarrow \varphi(\mathbf{g}x))$$

is in Γ .

- For each $k < \omega$ and each quantifier-free-complete k -type $q(x)$ of the language of MLO_n , for each formula $\varphi(x)$ of $\mathcal{L}_{\mathcal{M}}$ such that $\text{Th}(\mathcal{M})$ implies $\varphi \rightarrow X^k$, for all k -tuples a, b over the set of constants naming the countable model \mathcal{A}_n

$$q(a) \wedge q(b) \rightarrow (\varphi(\mathbf{f}ga) \leftrightarrow \varphi(\mathbf{f}gb))$$

is in Γ .

- The sentence,

$$(\forall z_0 \dots z_{n-1} \in A) (\forall x \in A) \bigwedge_{i < n} (x <_i z_i \leftrightarrow \psi(\mathbf{g}(x), \mathbf{h}_i(z)))$$

is in Γ .

- Γ asserts that \mathbf{f} is the mapping associated with a d -sub-interpretation using X , E and $(\varphi^I)_{\varphi}$ of the B in M .

- For each quantifier-free formula $\varphi(x_0, \dots, x_{k-1})$ of \mathcal{L} , the sentence

$$\forall x \left(\varphi(x) \leftrightarrow \exists v_0 \dots v_{k-1} \bigwedge_{i < k} E(v_i, \mathbf{f}x_i) \wedge \varphi^I(v) \right)$$

- Density: Suppose X_0, \dots, X_{N-1} are the cells of X , each with a definable bijection $e_i : X_i \rightarrow R_i$ onto a definable rectangle $R_i \subseteq M^{d_i}$ with $d_i \leq n$ ($i < N$, $d_i \leq n$)

$$\bigwedge_{i < N} (\forall x, y \in M^{d_i}) [\Pi_{\ell < N}(x_{\ell}, y_{\ell}) \subseteq R_i \rightarrow (\exists z \in B) e(\mathbf{f}(z)) \in \Pi_{\ell < N}(x_{\ell}, y_{\ell})]$$

□

This completes the proof of the theorem. □

An immediate consequence of the previous theorem (and Morleyzation) is the following, giving a loose characterization of any theory interpretable in an o-minimal theory as “stable in a sufficiently loose sense.”

Corollary 3.7. *Let \mathcal{M} be an o-minimal structure. For any structure \mathcal{A} , if \mathcal{A} is interpretable in a model of $\text{Th}(\mathcal{M})$, then for some $n < \omega$, $\text{Th}(\mathcal{A})$ does not have n -MOP.*

Recall that a structure is pseudo-o-minimal just in case it is elementarily equivalent to an ultraproduct of o-minimal structures, and a theory is pseudo-o-minimal just in case it has a pseudo-o-minimal model.

Proposition 3.8. *Let T be any op-minimal theory in a countable one-sorted language \mathcal{L} – i.e., $\text{opD}(T) \leq 1$. Then T is 1-dimensionally d -sub-definable in an pseudo-o-minimal structure.*

Proof. We fix a countable model \mathcal{B}_0 of T . We define a language \mathcal{L}^+ with two sorts X, Y and a function symbol $\mathbf{f} : X \rightarrow Y$; further, X carries the whole signature of \mathcal{L} , and for each formula $\varphi(x_0, \dots, x_{n-1})$ of \mathcal{L} (where the x_i s are single variables), let R_φ be an n -ary relation symbol on Y . Finally, let $0, 1$ be constant symbols on Y , and let $<$ be a binary relation symbol on Y . Let \mathcal{L}^{++} be the further expansion of \mathcal{L}^+ to include \mathbb{Q} as a set of constant symbols on Y (with $0, 1$ playing themselves). Let \mathcal{D} be the set of all pairs $(q_0, q_1) \in \mathbb{Q}^2$ such that $0 \leq q_0 < q_1 \leq 1$. For each $F \subset_{\text{fin}} \text{eddiag}(\mathcal{B}_0)$ and each $D \subset_{\text{fin}} \mathcal{D}$, let $\Sigma_{F,D}$ be the following set of sentences of \mathcal{L}^{++} :

- $\Sigma_{F,D}$ says $(Y, <) \models \text{DLO}$, and $(q_0 < q_1) \in \Sigma_{F,D}$ for all $(q_0, q_1) \in D$.
- $\Sigma_{F,D} \models “T \cup F \text{ on } X.”$
- $(\forall x \in X)(0 \leq \mathbf{f}(x) \leq 1)$ and “ \mathbf{f} is one-to-one” are in $\Sigma_{F,D}$.
- $\{R_\theta(\mathbf{f}(b_0), \dots, \mathbf{f}(b_{n-1})) : \theta(b_0, \dots, b_{n-1}) \in F\} \subseteq \Sigma_{F,D}$.
- For each $(q_0, q_1) \in D$, $(\exists x \in X)(q_0 < \mathbf{f}(x) < q_1)$ is in $\Sigma_{F,D}$.

Let $\mathcal{L}^{++}[F, D]$ be the sub-language of \mathcal{L}^{++} whose signature on the Y -part is restricted to

$$\{<\} \cup \mathbb{Q} \cup \{R_\varphi : \varphi \text{ is a sub-formula of a member of } F\}.$$

A subset $X \subseteq \mathcal{L}^{++}[F, D]$ is called a fragment of $\mathcal{L}^{++}[F, D]$ if it is closed under boolean combinations, changes of variables and taking sub-formulas.

Observation. Suppose that for any $F \subset_{\text{fin}} \text{eddiag}(\mathcal{B}_0)$ and $D \subset_{\text{fin}} \mathcal{D}$, there is a model $(\mathcal{B}_{F,D}, \mathcal{M}_{F,D}, f_{F,D})$ of $\Sigma_{F,D}$ in which $\mathcal{M}_{F,D}$ is a pseudo-o-minimal structure. Then T is 1-dimensionally pseudo-o-minimally d -sub-definable.

Proof of Observation. Let Ψ be any non-principal ultrafilter on the set $\mathcal{P}_{\text{fin}}(\text{eddiag}(\mathcal{B})) \times \mathcal{P}_{\text{fin}}(\mathcal{D})$ (where for any set X , $\mathcal{P}_{\text{fin}}(X)$ is the set of finite subsets of X), and let $(\mathcal{C}, \mathcal{M}, f) = \prod_{F,D} (\mathcal{B}_{F,D}, \mathcal{M}_{F,D}, f_{F,D}) / \Psi$. Clearly, $\mathcal{C} = \prod_{F,D} \mathcal{B}_{F,D} / \Psi$ is a model of T , and $\mathcal{M} = \prod_{F,D} \mathcal{M}_{F,D} / \Psi$ is a

pseudo-o-minimal expansion of $(M, <^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}})$. Moreover,

$$\left([0, 1], 1_{[0,1]}, (R_\varphi)_{\varphi \in \text{QF}(\mathcal{L})}, f \right)$$

is a d -sub-definition of \mathcal{B} in \mathcal{M} . \square

Claim. *Let $F \subset_{\text{fin}} \text{eddiag}(\mathcal{B}_0) \setminus T$, $\varphi \in \text{eddiag}(\mathcal{B}_0) \setminus T$, and $D \subset_{\text{fin}} \mathcal{D}$. Then $\Sigma_{F,D}$ has model $(\mathcal{B}, \mathcal{M}, f)$ such that \mathcal{M} is pseudo-o-minimal.*

Proof of claim. For a finite set $S \subset_{\text{fin}} \text{Sent}(\mathcal{L}^{++}[F, D])$, let us say that S is F, D -good if there is model $(\mathcal{B}, \mathcal{M}, f) \models \Sigma_{F,D} \cup S$ such that for any $\varphi(x, y) \in \mathcal{L}^{++}[F, D]$ with (x a single variable, x, y both of sort Y , and every sub-formula of φ in the language of \mathcal{M}), if $\varphi(x, y)$ is in the fragment of $\mathcal{L}^{++}[F, D]$ generated by $\Sigma_{F,D} \cup S$, then φ is *not* an obstruction to o-minimality of \mathcal{M} (meaning that there is a number $k < \omega$ such that for any $c \in M^y$, $\varphi(\mathcal{M}, c)$ is indeed equal to the union of $\leq k$ open intervals and $\leq k$ points).

Enumerating $\text{Sent}(\mathcal{L}^{++}[F, D])$ as $\{\varphi_j\}_{j < \omega}$, we define a tree $W \subseteq 2^{<\omega}$ consisting of those $\sigma \in 2^{<\omega}$ such that $\{\varphi_j^{\sigma(j)} : j < |\sigma|\}$ is F, D -good. As W is a finitely-branching tree, if it is infinite, then by König's Lemma, we recover an infinite branch $f : \omega \rightarrow 2$ of W , and this f encodes a complete pseudo-o-minimal theory $T_{F,D}$ such that

$$\Sigma_{F,D} \subseteq T_{F,D} \subseteq \text{Sent}(\mathcal{L}^{++}[F, D]).$$

For a contradiction, then, we assume that W is finite – in particular, $2^{<\omega} \setminus W$ has a finite set $\{\sigma_0, \dots, \sigma_{N-1}\}$ of minimal elements.

Let $b = (b_0, \dots, b_{m-1})$ and $q = (q_0, \dots, q_{m'-1})$ enumerate all of the elements of \mathcal{B}_0 and \mathbb{Q} , respectively, that appear in any $\varphi_j \in \sigma_i^{-1}(1)$, $i < N$. Without loss of generality, we may also assume that $|\sigma_i| = |\sigma_0| = \ell$ for each $i < N$. Moreover, by the Robinson Joint Consistency Theorem, for each $i < N$, the conjunction $\bigwedge_{j < \ell} \varphi_j^{\sigma_i(j)}$ is equivalent modulo $\Sigma_{F,D}$ to a conjunction $\bigwedge_{s < t_i} \psi_{i,s}$, where each $\psi_{i,s}$ is of the form

$$\eta_{i,s}(\mathbf{fb}, q) \wedge \theta_{i,s}(b) \rightarrow \theta'_{i,s}(\mathbf{fb})$$

and where $\eta_{i,s}$ is a quantifier-free-complete type in the language of order, $\theta_{i,s}$ is an \mathcal{L} -formula (the language of \mathcal{B}_0), and $\theta'_{i,s}$ is a formula of $\mathcal{L}^{++}[F, D]$ that has no X -sorted variables or constants at all and does not involve $<$.

Now, for any proper extension $\sigma_i \subset \tau \in 2^{<\omega}$ and any $(\mathcal{B}, \mathcal{M}, f) \models \Sigma_{F,D} \cup \left\{ \varphi_j^{\tau(j)} \right\}_{j < |\tau|}$, there are $i < N$, $s < t_i$, and $k < n$ such that the partitioned formula,

$$\theta'_{i,s}(y_k; y_0 \dots y_{k-1} y_{k+1} \dots y_{n-1})$$

is an obstruction to o-minimality in \mathcal{M} . For all $i < N$, $s < t_i$, $k < n$, this formula “pulls back” to an \mathcal{L} -formula, $\xi_{i,s}^k(v; w_1 \dots w_{n-1})$. (Let $\zeta_{i,s}^k(x; y_1 \dots y_{n-1})$ be the un-pulled-back formula.) We make an easily, if tediously, verifiable observation:

Observation. Suppose there is a finite model $(\mathcal{C}, <_0^{\mathcal{C}}, <_1^{\mathcal{C}}) \models \text{MLO}_2$ such that for any one-to-one map $g : C \rightarrow B_0$ and for each i_0, s_0, k_0 , there is multi-cut (X_0, X_1) such that for any i_1, s_1, k_1 , one cannot choose $b_0, b_1 \in B^{n-1}$, $\mathcal{B}_0 \preceq \mathcal{B}$, so that $X_j = \{c : \xi_{i_j, s_j}^{k_j}(g(c), b_j)\}$ for both $j = 0, 1$. Then there are numbers $e(i, s, k)$ such that

$\Sigma_{F,D} \cup \{\forall \bar{y} \text{ “}\zeta_{i,s}^k(x, \bar{y}) \text{ is the union of } \leq e(i, s, k) \text{ points and open intervals”}\}_{i,s,k}$.

is consistent.

Now, since $\text{opD}(T) \leq 1$, there must be such a finite model $(\mathcal{C}, <_0^{\mathcal{C}}, <_1^{\mathcal{C}})$ of MLO_2 ; otherwise, we would have $\text{opR}_2(\{x=x\}, \{\xi_{i,s}^k(v, \bar{w})\}_{i,s,k}) = \infty$. Consequently, we have a contradiction to the definition of $\sigma_0, \dots, \sigma_{N-1}$, so W must be infinite – which proves completes the proof of the claim and of the proposition. \square

\square

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(Guingona) UNIVERSITY OF NOTRE DAME, DEPARTMENT OF MATHEMATICS,
255 HURLEY, NOTRE DAME, IN 46556

E-mail address: guingona.1@nd.edu

URL: <http://www.nd.edu/~vguingon/>

(Hill) WESLEYAN UNIVERSITY, DEPARTMENT OF MATHEMATICS AND COM-
PUTER SCIENCE, 45 WYLLYS AVENUE, MIDDLETOWN, CT 06459

E-mail address: cdhill@wesleyan.edu