A Frobenius type theorem for sharply 2-transitive countable linear groups

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Abstract

Using the Glasner-Gelander theorem for primitive groups, we find a more precise classification for sharply 2-transitive groups.

Contents

1 Introduction

Some of the most appealing results in group theory are classification results. For finite groups many impressive achievements have been made, including the classification of finite simple groups (CFSG) and, following it, the classification of 2-transitive finite permutation groups. A much more modest result, and one that is independent of the CFSG, is the classification of finite sharply 2-transitive groups. For infinite groups, classification results were obtained only in much more restricted settings. One goal that does seem possible is the classification of countable sharply 2-transitive groups. Our motivation is to generalize some of the results known for finite sharply 2-transitive groups to the setting of countable linear groups.

By the famous theorem of Frobenius, there exists a splitting into a semi-direct product in finite sharply 2-transitive groups. Our method will extend this theorem to some sharply 2-transitive countable linear groups.

It is easy to see that every sharply 2-transitive group contains many involutions: an element of the group that flips a pair has to be of order 2, i.e. an involution. It turns out (see Proposition 2.6) that the order of the product of two distinct involutions is a group invariant, which will be called the *permutation characteristic* and denoted $p\text{-char}(\Gamma)$. By Corollary 7.6A in [DM], every finite sharply 2-transitive group G splits as a product $G = N^{\times} \ltimes N$ where N is a near-field. Recall that a near-field is an algebraic object that satisfies all the axioms of a field, except it has only left distributive law (see Definition 2.12). To generalize this result, we use the following theorem:

Theorem 1.1. *Let $\Gamma < \mathrm{GL}_n(F)$ be a sharply 2-transitive countable linear non-torsion group and $\Delta = \Gamma_{x_0}$ be the stabilizer of a point (we can assume that the linear representation is irreducible). Assume that $p\text{-char}(\Gamma) \neq 2$, and that $\mathrm{char}(F) \neq 2$* [1]*. Then one of the following holds:*

1. Γ **is permutation isomorphic to** $N^{\times} \ltimes N$ **where** N **is a near-field.**
2. **The action comes from a 2-transitive action of a simple algebraic group:** *After possibly changing the linear representation of Γ, and letting k be algebraic closure of F and $G, H < \mathrm{GL}_n(k)$ the Zariski closures of Γ and Δ respectively. Then the following holds:*
   (a) *G is close to being simple in the sense that $G^0 = S \times S \times S \times \ldots \times S$ is the direct power of some simple algebraic group and Γ acts transitively on these simple factors by conjugation.*
   (b) *There exists a nontrivial direct sum decomposition $k^n = V \oplus W$ such that*
   $$H < G_{\{V\}} \cap G_{\{W\}},$$
   where $G_{\{V\}} = \{g \in G \mid gV = V\}$ is the setwise stabilizer of V.
   (c) *The action of G on G/H is 2-transitive in the algebraic group sense. Namely G admits an open orbit in its action on $G/H \times G/H$.*

Conjecturally every sharply 2-transitive group splits as in Theorem 1.1(1) above. This work is an attempt to prove this conjecture for countable linear groups. In particular, we conjecture that case (2) of the theorem never occures. We prove this under few additional assumptions and conclude by sketching some attempts to rule out possibility of (2) for linear groups in positive permutation characteristic.

[1] All fields in this work will be assumed to be of field characteristic not equal to 2.

The proof of this theorem is based on the theorem of [GG] on classification of countable linear non-torsion primitive groups. By definition, a group is called "primitive" if it admits a faithful primitive action on a set. Thus, in particular, every sharply 2-transitive group is primitive (see Lemma 2.2).

We now turn to establish additional restrictions on sharply two transitive groups of positive permutational characteristic. The first theorem in this direction is (Same as 4.4 below)

Theorem 1.2. *If* Γ *is an infinite sharply* 2*-transitive group,* M *is a compact metric space and* $\Gamma \curvearrowright M$ *is a non-elementary convergence action, then* $p\text{-char}(\Gamma) = 0$.

In particular the above theorem applies to Fuchsian groups, Gromov hyperbolic groups and relatively hyperbolic groups. Indeed, the proof relies on dynamics of the action of these groups on their natural boundaries.

After proving Theorem 1.2 we describe some of our attempts to use similar dynamical methods in order to rule out the second case (2a) of Theorem 1.1 when Γ is of positive permutational characteristic. Indeed it is well known that non-solvable linear groups admit representations over valued fields with a rich boundary dynamics; for a precise statement see Theorem 4.16. In particular this applies to groups of the second type in the main Theorem and thus the dynamical methods used in the proof of Theorem 1.2 may be applicable in this case. A precise result in this direction is stated as Theorem 4.18

2 Preliminaries

2.1 Terminology

Let Ω be a countable set and $\mathrm{Sym}(\Omega)$ be the full group of permutations.

Definition 2.1. A permutation group $\Gamma < \mathrm{Sym}(\Omega)$ is called

- *Primitive* if there is no non-trivial Γ-invariant equivalence relation on Ω. (i.e. if $\sim$ is an equivalence relation on Ω s.t. $\forall a, b \in \Omega$ we have $a \sim b \Rightarrow \gamma a \sim \gamma b \ \forall \gamma \in \Gamma$ then the equivalence classes are either singletons or the whole of Ω)
  Alternatively, Γ is primitive if it is transitive and the point stabilizer, $\Gamma_a = \{\gamma \in \Gamma | \gamma a = a\}$, is a maximal subgroup.
- *2-transitive* if it acts transitively on ordered pairs of distinct points.
- *Sharply 2-transitive* if it is 2-transitive and the stabilizer of an ordered pair is trivial.

These definitions are arranged from the weaker to the stronger: clearly, every sharply 2-transitive group is 2-transitive, and

Lemma 2.2. *Every* 2*-transitive group is primitive.*

Proof. Let $\sim$ be a Γ-invariant equivalence relation on Ω. Assume, for contradiction, that $\sim$ is non-trivial. Then there exists $a, b, c \in \Omega$ such that $a \neq b$, $a \sim b$ and $a \nsim c$. By the Γ-invariance of $\sim$, we have $\gamma a \sim \gamma b$ for all $\gamma \in \Gamma$. On the other hand, by the 2-transitivity assumption, there exists $\delta \in \Gamma$ such that $\delta a = a$ and $\delta b = c$, but then $\delta a \nsim \delta b$, a contradiction. □

Observe that if Γ is a sharply 2-transitive group then, from the 2-transitivity assumption, for all $a \neq b \in \Omega$ there exists $\sigma \in \Gamma$ such that $\sigma a = b$ and $\sigma b = a$. Hence $\sigma^2 a = a$ and $\sigma^2 b = b$ and, from the "sharply" assumption, $\sigma^2 = 1$. So every sharply 2-transitive group contains many elements of order 2. Such elements will be of a particular interest to us.

Definition 2.3. Let Γ be a group.

- An element $\sigma \in \Gamma$ is an *involution* if $\mathrm{Ord}(\sigma) = 2$. (i.e. $\sigma \neq 1$ and $\sigma^2 = 1$)
- The set of all involutions in Γ will be denoted by $\mathrm{Inv}(\Gamma)$.
- We define $\mathrm{Inv}(\Gamma)^2 = \{\sigma\tau | \sigma, \tau \in \mathrm{Inv}(\Gamma)\}$

Note that if σ is an involution then so is $\sigma^\gamma = \gamma^{-1}\sigma\gamma$ for all $\gamma \in \Gamma$, hence $\mathrm{Inv}(\Gamma)$ is closed under conjugation. Moreover, $\mathrm{Inv}(\Gamma)$ is a conjugacy class in Γ:

Lemma 2.4. *If Γ is a sharply 2-transitive group then $\mathrm{Inv}(\Gamma)$ is a conjugacy class in Γ.*

Proof. Let $\sigma, \tau \in \mathrm{Inv}(\Gamma)$. An involution is uniquely determined by any pair of points it flips. Assume σ flips x, y and τ flips z, w. By 2-transitivity, there exists $\gamma \in \Gamma$ such that $\gamma z = x$ and $\gamma w = y$. Then σ^γ is an involution and $\sigma^\gamma z = w$, hence $\sigma^\gamma = \tau$. □

As a conclusion from this lemma we get that either every involution stabilizes a point or none does. Moreover:

Lemma 2.5. *If Γ is a sharply 2-transitive group then Γ_x contains at most one involution.*

Proof. Assume $\sigma, \tau \in \Gamma_x$ are involutions. Fix any $y \neq x$. Since Γ is sharply 2-transitive, there exists $\gamma \in \Gamma_y$ such that $\gamma(\tau y) = \sigma y$. Then we have $\sigma^\gamma y = \gamma^{-1}\sigma y = \tau y$. By the same argument as before, we have $\sigma^\gamma = \tau$, which implies $x = \tau x = \gamma^{-1}\sigma\gamma x$. Thus $\sigma(\gamma x) = \gamma x$. Hence $\sigma \in \Gamma_x \cap \Gamma_{\gamma x}$. By the "sharply" assumption, this implies $\gamma x = x$. Thus γ fixes both x, y, which means that $\gamma = 1$. □

More interesting is the fact that $\mathrm{Inv}(\Gamma)^2 \setminus \{1\}$ is a conjugacy class, which, in turn, accounts for the following proposition that will be important to us.

Proposition 2.6. *Let Γ be a sharply 2-transitive group and assume that Γ_x contains an involution. Then all elements in $\mathrm{Inv}(\Gamma)^2 \setminus \{1\}$ share the same order which is either ∞ or a prime number p.*

Definition 2.7. Let $\Gamma \leq \mathrm{Sym}(\Omega)$. An element $\gamma \in \Gamma$ is called *fixed-point free* if it does not fix a point. A subgroup $H \leq \Gamma$ is called *fixed-point free* if all elements in $H \setminus \{1\}$ are fixed-point free.

Lemma 2.8. *If Γ is a sharply 2-transitive group then all elements in $\mathrm{Inv}(\Gamma)^2 \setminus \{1\}$ are fixed-point free.*

Proof. Let $\sigma, \tau \in \mathrm{Inv}(\Gamma)$. Assume that there exists $x \in \Omega$ such that $\sigma\tau \in \Gamma_x$, i.e $\sigma x = \tau x$. Hence:

Either

- $\sigma x = \tau x = x$. Then $\sigma, \tau \in \Gamma_x$ and by Lemma 2.5, $\sigma = \tau$

Or

- $\sigma x = \tau x = y \neq x$. Then σ, τ coincide on two distinct points, x and y, hence are equal. □

Proposition 2.9. *Let Γ be a sharply 2-transitive group and assume Γ_x contains an involution. Then the given sharply 2-transitive action of Γ is isomorphic to the action of Γ on $\mathrm{Inv}(\Gamma)$ by conjugation.*

Proof. Let $\Gamma \curvearrowright X$ be a sharply 2-transitive action. First, by Lemma 2.4, the action $\Gamma \curvearrowright \mathrm{Inv}(\Gamma)$ is transitive. Now, it's sufficient to show that the stabilizers of points in both actions are equal, i.e $\forall x \in X$ $\exists \sigma \in \mathrm{Inv}(\Gamma)$ such that $\Gamma_x = \Gamma_\sigma$, and vice versa. Take any $x \in X$. By assumption on Γ and by Lemma 2.5, the stabilizer contains a unique involution σ which is centralized by Γ_x. Since Γ_x is a maximal subgroup, we have $\Gamma_x = \Gamma_\sigma$. □

Proof of Proposition 2.6. We will prove this proposition in two steps:

(I) $\mathrm{Inv}(\Gamma)^2 \setminus \{1\}$ is one conjugacy class: Follows immediately from Proposition 2.9.

(II) $\mathrm{Inv}(\Gamma)^2$ is closed under powers: Take any $\sigma\tau \in \mathrm{Inv}(\Gamma)^2 \setminus \{1\}$ and $m \in \mathbb{N}$.

- If m is odd then $(\sigma\tau)^m = (\sigma\tau)^{\frac{m-1}{2}} \sigma (\tau\sigma)^{\frac{m-1}{2}} \tau$ is once again a product of two involutions.
- If m is even then $(\sigma\tau)^m = \left((\sigma\tau)^{\frac{m}{2}-1}\sigma\right) \tau \left(\sigma(\tau\sigma)^{\frac{m}{2}-1}\right) \tau$ is a product of two involutions.

From (I) we know that all elements in $\mathrm{Inv}(\Gamma)^2 \setminus \{1\}$ share the same order. If that order is finite, denote it by n. Assume, by contradiction, that $n = p \cdot q$ is composite ($p, q \neq 1$). Then, by (II), $(\sigma\tau)^q$ is another element in $\mathrm{Inv}(\Gamma)^2 \setminus \{1\}$, whose order is $p \neq n$, which is a contradiction. □

The previous proposition leads us to the definition of the following invariant:

Definition 2.10. Let Γ be a sharply 2-transitive group. The *permutation characteristic* of Γ, denoted by $p\text{-char}(\Gamma)$, is defined as follows:

- If Γ_x does not contain an involution then define $p\text{-char}(\Gamma) = 2$.
- If Γ_x contains an involution then:
  - ⋄ $p\text{-char}(\Gamma) = 0 \Leftrightarrow$ The order of all elements in $\mathrm{Inv}(\Gamma)^2 \setminus \{1\}$ is ∞.
  - ⋄ $p\text{-char}(\Gamma) = p \Leftrightarrow$ The order of all elements in $\mathrm{Inv}(\Gamma)^2 \setminus \{1\}$ is the prime number p.

Example 2.11. Let D be a division ring and let $\Gamma = D^\times \ltimes D$ with the multiplication defined by $(a, b) \cdot (a', b') = (aa', ab' + b)$. Observe the action $\Gamma \curvearrowright D$ defined by $(a, b)x = ax + b$. This action is sharply 2-transitive since for every two pairs of distinct points $x \neq y, z \neq w$ in D we can find a unique element $(a, b) \in \Gamma$ such that $ax + b = z$ and $ay + b = w$ by solving a system of two linear equations. Now, let us describe $\mathrm{Inv}(\Gamma)$: $(a, b) \in \mathrm{Inv}(\Gamma) \Leftrightarrow (a, b)^2 = (a^2, ab + b) = (1, 0)$. Hence $a \in \{1, -1\}$.

- If $a = 1$ then $2b = 0$. Hence:
  - ⋄ if $\mathrm{char}(D) = 2$ then any $b \neq 0$ will give us a non-trivial involution.
  - ⋄ if $\mathrm{char}(D) \neq 2$ then $b = 0$, which gives us the identity.
- If $a = -1$ then any b will give us a non-trivial involution (except for $b = 0$ if $\mathrm{char}(D) = 2$).

So $\mathrm{Inv}(\Gamma) = \{(-1, b) | b \in D\}$. Now take the product of two distinct involutions: $(-1, b) \cdot (-1, c) = (1, b - c)$. Hence $((-1, b) \cdot (-1, c))^n = (1, b - c)^n = (1, n(b - c))$ which is equal to $(1, 0)$ if and only if $n(b - c) = 0$. Since $b \neq c$, we must have $n = 0$, which implies that $p\text{-char}(\Gamma) = \mathrm{char}(D)$

2.2 Near-Fields

Observing Example 2.11 more closely, one can see that the action $\Gamma = D^\times \ltimes D \curvearrowright D$ is sharply 2-transitive, even if D satisfies only a left distributive law. In other words, if D is a near-field.

Definition 2.12. A 5-tuple $\langle N, +, \cdot, 0, 1\rangle$ is called a *near-field* if it satisfies the following 4 axioms:

NF1: $\langle N, +, 0\rangle$ is an Abelian group.

NF2: $\langle N \setminus \{0\}, \cdot, 1\rangle$ is a group.

NF3: For all $a, b, c \in N$ we have $a \cdot (b + c) = a \cdot b + a \cdot c$.

NF4: $a \cdot 0 = 0 \cdot a = 0$ For all $a \in N$.

Remark This structure is sometimes called a left near-field. One can define a right near-field by replacing **NF3** with right distributivity.

One can check that if F is a near-field, then the group $F^\times \ltimes F$ admits a sharply 2-transitive action on F, as defined in Example 2.11.

Theorem 2.13 (Theorem 7.6C in [DM]). *Let $|\Omega| \geq 2$ and let $G \leq \mathrm{Sym}(\Omega)$ be a sharply 2-transitive group which possesses a fixed-point free normal abelian subgroup K. Then there exists a near-field N such that G is permutation isomorphic to $N^\times \ltimes N$.*

For the convenience of the reader, we shall add a sketch of the proof:

Proof. Set $N = \Omega$. Fix two arbitrary elements in N and denote them by $0, 1$. Since K is fixed-point free, the map $k \mapsto k(0)$ defines a bijection between K and N. Define addition on N by $k(0)+m(0) = km(0)$, which turns the map above to an isomorphism from K to $\langle N, +\rangle$. Clearly, **NF1** holds.
Since G_0 is fixed-point free on $\Omega \setminus \{0\}$, now using the map $g \mapsto g(1)$, we can define multiplication on $N^\times = N \setminus \{0\}$ by $g(1) \cdot h(1) = gh(1)$. With this definition, G_0 and $\langle N^\times, \cdot\rangle$ are isomorphic, which implies **NF2**.
Define $a \cdot 0 = 0 \cdot a = 0$ For all $a \in N$. This guarantees **NF4**.
K is normal, so we have $k^g \in K$ for all $k \in K$ and $g \in G_0$. Also, $g(1) \cdot k(0) = k^{g^{-1}}(0)$:

- If $k = 1_G$ then $k^g = 1_G$ which implies $g(1) \cdot k(0) = g(1) \cdot 0 = 0 = k^{g^{-1}}(0)$.
- If $k \neq 1_G$ then there exists a unique $h \in G_0$ such that $k(0) = h(1)$. Then $g(1) \cdot k(0) = g(1) \cdot h(1) = gh(1) = gk(0) = k^{g^{-1}}(0)$, since $g \in G_0$.

Now take any $a, b, c \in N$.

- ∘ If $a = 0$ then $0 \cdot (b + c) = 0 = 0 + 0 = 0 \cdot b + 0 \cdot c$.
- ∘ If $a \neq 0$ then $a = g(1), b = k(0), c = m(0)$ where $g \in G_0$ and $k, m \in K$. We have

$$a \cdot (b + c) = g(1) \cdot (k(0) + m(0)) = g(1) \cdot km(0) = (km)^{g^{-1}}(0) = k^{g^{-1}} m^{g^{-1}}(0)$$

$$= k^{g^{-1}}(0) + m^{g^{-1}}(0) = g(1) \cdot k(0) + g(1) \cdot m(0) = a \cdot b + a \cdot c$$

This proves **NF3**, and completes the proof that N is a near-field. Moreover:

- ⊙ K acts on N by: $k(a) = km(0) = k(0) + m(0) = a + k(0)$.
- ⊙ G_0 acts on N by: $g(a) = gh(1) = g(1) \cdot h(1) = g(1) \cdot a$. (If $a = 0$ then $g(0) = 0 = g(1) \cdot 0$).

Since K is a fixed-point free normal subgroup, every $\gamma \in G$ can be written as $\gamma = gk$ for some $g \in G_0$ and $k \in K$. Hence we have

$$\gamma(a) = gk(a) = g(a + k(0)) = g(1) \cdot (a + k(0)) = g(1) \cdot a + g(1) \cdot k(0)$$

So G acts on N as $x \mapsto a \cdot x + b$ for $a \in N \setminus \{0\}$ and $b \in N$. Since G is 2-transitive, it contains all the permutations of this form. □

2.3 The Gelander-Glasner theorem

Our current result is based on a previous theorem due to Gelander and Glasner (see [GG]) on the structure of primitive countable non-torsion linear groups, i.e. such groups that admit a faithful primitive action on a set. Before quoting this theorem, which is somewhat similar to the Aschbacher-O'Nan-Scott theorem for finite primitive groups, let us highlight some classes of primitive permutation groups:

Definition 2.14. Suppose V is vector space over a prime field F and let $\Delta < \mathrm{GL}(V)$. Assume that there are no Δ invariant additive subgroups in V. Then $\Gamma = \Delta \ltimes V$ admits a primitive action on V, where V acts on itself by translations and Δ acts by the given action, i.e. $(\delta, v)(w) = \delta(w) + v$. In this case, we say that Γ is *primitive of affine type*. Such an action of a semi direct product is called an *affine action*.

Lemma 2.15. *The action of* $\Gamma = \Delta \ltimes V$ *on* V *as described above is primitive.*

Proof. Assume, for contradiction, that there exists a non-trivial Γ-invariant equivalence relation on V, denoted by $\sim$. Let $W = \{v \in V | v \sim 0\} = [0]$.

- Observe that W is an additive subgroup of V: Let $v, w \in W$ then we have $(1_\Delta, v)w = w + v$ by definition. On the other hand, $(1_\Delta, v)w \sim (1_\Delta, v)0 = 0 + v = v$ since $\sim$ is Γ-invariant. Hence $w + v \in W$. Moreover, if $w \in W$ then $0 = (1_\Delta, -w)w \sim (1_\Delta, -w)0 = -w$. Clearly $0 \in W$.
- $W \neq \{0\}, V$ since $\sim$ is non-trivial.
- Furthermore, W is Δ-invariant since for all $\delta \in \Delta$ and $w \in W$ we have

$$\delta w = (\delta, 0)w \sim (\delta, 0)0 = \delta 0 = 0$$

This construction gives us a Δ-invariant proper additive subgroup in V, which is a contradiction. □

Definition 2.16. Suppose M is nonabelian characteristically simple group, and let $\mathrm{Inn}(M) < \Delta < \mathrm{Aut}(M)$. Assume that there are no Δ invariant subgroups in M. Then $\Gamma = \Delta \ltimes M$ admits an affine action on M, as above. In this case, we say that Γ is *primitive of diagonal type*.

Remark Same argument as in Lemma 2.15 shows that this action is also primitive.

Definition 2.17. A group Γ is *primitive of the almost simple type* if it has a faithful representation $\rho : \Gamma \to \mathrm{GL}_n(K)$, where K is an algebraically closed field, with Zariski closure $\mathbb{G} = \overline{\rho(\Gamma)}^Z$, s.t:

1. The identity connected component of $\mathbb{G}$, $\mathbb{G}^0$, decomposes into direct product of simple factors with trivial centers.
2. The action $\Gamma \curvearrowright \mathbb{G}^0$ by conjugation is faithful and permutes the simple factors transitively.

Unlike the previous two examples, there is no apparent primitive action of such a group. In fact, the most difficult result of [GG] is that such a group always admits a faithful primitive action. In fact:

Theorem 2.18. Theorem 1.9 in [GG] *A countable non torsion linear group* Γ *is primitive (see Definition 2.1) if and only if one of the following mutually exclusive conditions hold:*

- Γ *is primitive of affine type.*
- Γ *is primitive of diagonal type.*
- Γ *is primitive of the almost simple type.*

2.4 Linear Groups

Let us recall three important results:

Corollary 2.19 (Corollary 3.8 from [W], due to Zassenhaus)**.** *A locally soluble linear group is soluble. Every linear group contains a unique maximal soluble normal subgroup.*

Corollary 2.20 (Corollary 4.9 from [W], due to Schur)**.** *A periodic linear group is locally finite.*

Theorem 2.21 (Engel's Theorem, Section 1.2 in [R])**.** *Any Lie subgroup of* $\mathrm{GL}(V)$ *consisting of unipotent automorphims of* V *has an invariant vector* $v \neq 0$*.*

Using those results we can prove the following:

Proposition 2.22. *Let* $\Bbbk$ *be an algebraically closed field and let* $G < \mathrm{GL}_n(\Bbbk)$ *be an infinite subgroup such that all its non-identity elements are conjugate, i.e.* $\forall A, B \in G \setminus \{I\} \;\; \exists p \in \mathrm{GL}_n(\Bbbk)$ *such that* $B = pAp^{-1}$*. Then* G *is soluble.*

Proof of Proposition 2.22.
If G is periodic, then all elements of G have the same order, which is a prime number p. By Corollary 2.20, G is locally finite. Hence every finitely generated subgroup $H \leq G$ is a finite p-group. A finite p-group is nilpotent and so it is soluble. Thus G is a locally soluble linear group. By Corollary 2.19, G is soluble.
If there exists an element $A \in G \setminus \{I\}$ of infinite order, then fix it. By assumption on Γ, $A^k \sim A$ for all $k > 0$. Thus, if $\{\lambda_1, ..., \lambda_m\}$ is the set of distinct eigenvalues of A, we know that $\{\lambda_1^k, ..., \lambda_m^k\} = \{\lambda_1, ..., \lambda_m\}$ for all $k \in \mathbb{N}$. Fix any $1 \leq j \leq m$. Since $\{\lambda_j, \lambda_j^2, ..., \lambda_j^{m+1}\} \subseteq \{\lambda_1, ..., \lambda_m\}$, if $\lambda_j^{k_1} \neq \lambda_j^{k_2}$ for all $1 \leq k_1 < k_2 \leq m+1$ then $\left|\{\lambda_j, \lambda_j^2, ..., \lambda_j^{m+1}\}\right| = m+1$, while $|\{\lambda_1, ..., \lambda_m\}| = m$, a contradiction. So there exist $1 \leq k_1 < k_2 \leq m+1$ such that $\lambda_j^{k_1} = \lambda_j^{k_2}$, thus $\lambda_j^{k_2 - k_1} = 1$, and $k_2 - k_1 > 0$ (of course, $\lambda_j \neq 0$ since A is invertible). So, for every $1 \leq j \leq m$ we found $k_j > 0$ such that $\lambda_j^{k_j} = 1$. Denote $K = k_1 \cdot k_2 \cdot ... \cdot k_m$. Now we have $\lambda_j^K = 1$ for all $1 \leq j \leq m$, so $\{1\} = \{\lambda_1^K, ..., \lambda_m^K\} = \{\lambda_1, ..., \lambda_m\}$. Now we know that all eigenvalues of A are 1, thus the characteristic polynomial of A is $P_A(t) = (t-1)^n$. By Cayley Hamilton theorem, $(A - I)^n = 0$. Since all $B \in G \setminus \{I\}$ are conjugate to A, we have $P_B(t) = (t-1)^n$ for all $B \in G \setminus \{I\}$. From this, for all elements $X \in G$ the equation $(X - I)^n = 0$ holds.
Take the Zariski closure $\overline{G}^Z$ of G in $\mathrm{GL}_n(\Bbbk)$. From a the above, $\overline{G}^Z \subseteq \{X \in \mathrm{GL}_n(\Bbbk) | \, (X - I)^n = 0\}$, since it is a polynomial equation. Thus $\overline{G}^Z$ is a Lie group with all unipotent elements, i.e. a unipotent group. By Theorem 2.21, inductively, we can bring $\overline{G}^Z$ simultaniusly to upper trinagular form with 1 on the diagonal. The last group is nilpotent, hence so is $\overline{G}^Z$. Since $G < \overline{G}^Z$, G is nilpotent and hence soluble. ☐

Theorem 2.23. *If* Γ *is a* 2*-transitive linear group then* Γ *is not primitive of the diagonal type (see Definition 2.16)*

Proof. Assume, for contradiction, that $\Gamma = \Gamma_{x_0} \ltimes M$, where M is a nonabelian characteristically simple subgroup and $\Gamma_{x_0} \curvearrowright M$ by conjugation. Since Γ is sharply 2-transitive, the stabilizer of each point, in particular Γ_{x_0}, is transitive on $M \setminus \{e\}$, thus every two elements of $M \setminus \{e\}$ are conjugate. Applying Proposition 2.22 on M we get that M is soluble. But then $[M, M]$ is a characteristic subgroup, thus is either M or $\langle e \rangle$. M is soluble, thus $[M, M] \subsetneqq M$, so $[M, M] = \langle e \rangle$, but then M is abelian - a contradiction. ☐

3 Proof of the main theorem

Proof of Theorem 1.1.
Γ is a sharply 2-transitive countable linear group and p-char$(\Gamma) \neq 2$. By Lemma 2.2 Γ is primitive. By the Gelander-Glasner theorem (2.18) Γ is either primitive of the almost simple type or primitive of diagonal type or primitive of affine type.
By Theorem 2.23, Γ is not primitive of the diagonal type. If Γ is primitive of affine type then, by definition, we have $\Gamma = \Delta \ltimes V$, where V is a vector space. Now we can apply Theorem 2.13 on Γ. Hence Γ is permutation isomorphic to $N^{\times} \ltimes N$ where N is a near-field.
So assume now that Γ is primitive of the almost simple type. The existence of a representation satisfying the conditions of (2a) is just the definition of primitive groups of almost simple type.

By our assumption on the permutational characteristic of Γ the group Δ contains a unique involution σ, which is hence centralized by all elements of Δ. Set $F^n = v \oplus w$ be the decomposition into the ± 1 eigenspaces of the involution σ. Since σ is central in Δ each element of Δ stabilizes both eigenspaces. Namely $\Delta < \Gamma_{\{v\}} \cap \Gamma_{\{w\}}$. But Δ is maximal in Γ so we have:

$$\Delta = \Gamma_{\{v\}} = \Gamma_{\{w\}}.$$

Since we assumed the irreducibility of the action, $\Gamma_{\{v\}} \neq \Gamma$. But stabilizing a subspace is a Zariski closed condition so if we set $V = v \otimes_F k, W = w \otimes_F k$ we will have automatically that $H < G_{\{V\}} \cap G_{\{W\}}$. Thus proving (2b)

Let $\gamma \in \Gamma \setminus \Delta$ be any element not in Δ. By definition of 2-transitivity we have $\Gamma/\Delta = \Delta \sqcup \Delta\gamma\Delta$. Now consider the set $H \sqcup H\gamma H < G/H$. This is a Zariski dense set since it contains the dense set $\Gamma H = H \sqcup \Delta\gamma H$ (which is dense since $\bar{\Gamma}^Z = G$); moreover it is locally closed (i.e. open in its closure, see [Hu, Proposition 8.3]) since it is an orbit of an algebraic action. Thus this set is open in G/H and hence $G(H, \gamma H) < G/H \times G/H$ is open in $G/H \times G/H$ proving (2c). □

4 Dynamical arguments in non-zero permutation characteristic

4.1 Convergence groups

Definition 4.1. Let M be a compact metric space and let Homeo(M) denote the set of all self homeomorphisms of M.

- An infinite set $H \subset$ Homeo(M), where M is an infinite compact space, is called *collapsing* with respect to a pair of points (a, r) if for every pair of compact sets $K \subset M \setminus \{r\}$ and $L \subset M \setminus \{a\}$, the set $\{h \in H | \ hK \bigcap L \neq \emptyset\}$ is finite.

- An action of an infinite group Γ on a compact topological space M has the *convergence property* if every infinite subset $H \subset \Gamma$ contains a further infinite subset $H' \subset H$ which is collapsing with respect to some pair of points.

- An action of an infinite group Γ on a compact topological space M that has the convergence property is called an *elementary convergence action* if Γ has a fixed point in M or if there are two points which are fixed as a pair.

- A group Γ is called a *convergence group* if it admits an action with the convergence property on some infinite compact Hausdorff space.

- A convergence group Γ is called *non-elementary* if it admits a non-elementary convergence action.

We will be interested in non-elementary convergence groups since every topological group admits an elementary convergence action on the one-point compactification of itself by (left or right) multiplication, fixing the added point.

Example 4.2.

I. Hyperbolic groups, in their action on the boundary of the hyperbolic space (see [Fr]).

II. Relatively hyperbolic groups can also be realized as convergence groups (see [Y]).

Proposition 4.3. *Let Γ be a group and let $\Gamma \curvearrowright M$ be a convergence action. Let $\Phi \subset \mathrm{Inv}(\Gamma)$ be an infinite set of involutions which is collapsing with respect to (a, r). Then $a = r$.*

Proof. Assume, by contradiction, that $a \neq r$. Since M is Hausdorff, there exist $a \in A, r \in R$ disjoint open neighborhoods. Then $M \setminus A, M \setminus R$ are compact sets (since they are closed in a compact Hausdorff space). Moreover, since M is infinite, it is possible to choose A, R such that $A \cup R \neq M$. Using the collapsing property:

$$\{\varphi \in \Phi | \ \varphi(M \setminus R) \cap (M \setminus A) \neq \emptyset\} = \Phi \setminus \{\varphi \in \Phi | \ \varphi(M \setminus R) \cap (M \setminus A) = \emptyset\} = \Phi \setminus \{\varphi \in \Phi | \ \varphi(M \setminus R) \subset A\}$$

is finite. Fix one $\varphi \in \Phi$ such that $\varphi(M \setminus R) \subset A$. Applying φ to both sides and recalling that $\varphi^2 = 1$, we get:

$$M \setminus R = \varphi^2(M \setminus R) \subset \varphi A \subset \varphi(M \setminus R) \subset A$$

(since $A \subset M \setminus R$). Which implies $A = M \setminus R$, contradicting our choice of A, R. □

Theorem 4.4. *If Γ is an infinite sharply 2-transitive group and $\Gamma \curvearrowright M$ is a non-elementary convergence action, then $p\text{-char}(\Gamma) = 0$.*

Proof. By Proposition 4.3, there exists a collapsing subset of involutions, $\{\sigma_i | i \in \mathbb{N}\}$, with respect to one point, $a \in X$. Since the action is non-elementary, there exists $\gamma \in \Gamma$ such that $\gamma a \neq a$. Then the set $\{\sigma_n^\gamma = \gamma^{-1}\sigma_i\gamma | i \in \mathbb{N}\}$ is collapsing with respect to $\gamma^{-1}a$. Using the Hausdorff property, take $a \in U$, $\gamma^{-1}a \in V$ disjoint open neighborhoods. Since M is infinite, it is possible to choose U, V such that $U \cup V \neq M$. $M \setminus U$ is closed, hence compact. By the collapsing property, there exists N_1 such that for all $n > N_1$, $\sigma_n(M \setminus U) \subset U$. For the same reason, there exists N_2 such that for all $n > N_2$, $\sigma_n^\gamma(M \setminus V) \subset V$. Denote $N = \max\{N_1, N_2\}$.
For all $n > N$, $\sigma_n(M \setminus U) \subset U$ and $\sigma_n^\gamma(M \setminus V) \subset V$. Take any $x \in M \setminus (U \cup V)$, then

$$\sigma_n\sigma_n^\gamma x \in \sigma_n\sigma_n^\gamma(M \setminus V) \subset \sigma_n V \subset \sigma_n(M \setminus U) \subset U$$

Similarly, $\sigma_n\sigma_n^\gamma U \subset \sigma_n\sigma_n^\gamma(M \setminus V) \subset U$. Hence, inductively, for all $k > 0$, $(\sigma_n\sigma_n^\gamma)^k x \in U$, in particular, $(\sigma_n\sigma_n^\gamma)^k x \neq x$. Thus $\sigma_n\sigma_n^\gamma$ is of infinite order. So, by definition, $p\text{-char}(\Gamma) = 0$. □

4.2 Proximality

Definition 4.5.

- An *absolute value* on a field F is a real-valued function $|\cdot| : F \to \mathbb{R}$ satisfying the following three properties:

- ▶ $|x| \geq 0$ for all $x \in F$ and $|x| = 0$ if and only if $x = 0$.
- ▶ $|xy| = |x| \cdot |y|$ for all $x, y \in F$.
- ▶ $|x + y| \leq |x| + |y|$ for all $x, y \in F$.

- A *valued field* is a field F equipped with an absolute value $|\cdot| : F \to \mathbb{R}$.
- An absolute valuem $|\cdot|$ is *non-Archimedean* if it satisfies the ultrametric triangle inequality: for any $x, y \in F$ we have $|x + y| \leq \max\{|x|, |y|\}$.
- Given a vector space V over a field F, define an equivalence relation $\sim$ on V by $v \sim w$ if there exists $x \in F^\times$ such that $v = x \cdot w$. The space of representatives, $V/\sim$, is called the *projective space* and denoted by $\mathbf{P}(V)$.

In this section we will restrict our attention to finite dimensional vector space V over a valued field F. The norm on F gives rise to a norm on V: fix any basis $\{v_i\}_{i=1}^n$, represent every vector as $v = \sum_{i=1}^n a_i v_i$. Then we can define $\|v\|_2 = \sqrt{|a_1|^2 + \ldots + |a_n|^2}$ if F is Archimedean and $\|v\|_\infty = \max\{|a_1|, \ldots, |a_n|\}$ if it is not. Similarly, using $\{v_i \wedge v_j\}_{1 \leq i < j \leq n}$ as a basis to the exterior power $V \wedge V$, we get a norm on this space. Now we can define a metric on $\mathbf{P}(V)$: $d([v], [w]) = \frac{\|v \wedge w\|}{\|v\|\|w\|}$. Two interesting properties of this metric are the following:

Proposition 4.6. *Let V be a finite dimensional vector space over a valued field F. For every linear transformation $T : V \to V'$ Define $\|T\| = \sup\{\|Tu\| \mid \|u\| = 1\}$, the induced transformation norm.*

I) Let $f : V \to F$ be a linear functional. Then for every $v \in V \setminus \{0\}$ we have $d([v], \mathbf{P}(\ker(f))) = \frac{|fv|}{\|v\|\|f\|}$.

II) Let $T \neq 0$ be a singular operator on V. Then there exists a constant $C > 0$ such that for all $\varepsilon > 0$ and $v \in V \setminus \{0\}$ if $d([v], \mathbf{P}(\ker(T))) > \varepsilon$ then $\|Tv\| > C\varepsilon$.

Proof. Let $\langle \cdot, \cdot \rangle$ be an appropriate inner product on V. Fix $v \in V \setminus \{0\}$.

I) Denote $W = (\ker(f))^\perp$ and let P be the orthogonal projection on W. Clearly, $d([v], \mathbf{P}(\ker(f))) = d([v], [v - Pv])$. Also,

$$d([v], [v - Pv]) = \frac{\|(Pv + v - Pv) \wedge (v - Pv)\|}{\|v\| \, \|v - Pv\|} = \frac{\|Pv \wedge (v - Pv)\|}{\|v\| \, \|v - Pv\|} = \frac{\|Pv\| \, \|v - Pv\|}{\|v\| \, \|v - Pv\|} = \frac{\|Pv\|}{\|v\|}$$

Observe that $Pv = \frac{f(v)}{\|f\|^2} f^t$, where f^t is transposed vector of f, after selecting a basis: indeed, since $W = \text{Span}\{f^t\}$, we should have $Pv = \alpha f^t$ and $f(v) = f(Pv) = \alpha f(f^t) = \alpha \|f\|^2$, which implies $\alpha = \frac{f(v)}{\|f\|^2}$. It follows that $\frac{\|Pv\|}{\|v\|} = \frac{|fv|}{\|v\|\|f\|}$.

II) Denote $W = (\ker(T))^\perp$ and let P be the orthogonal projection on W. As in I), we have $d([v], \mathbf{P}(\ker(T))) = \frac{\|Pv\|}{\|v\|}$. The set $S_W^1 = \{w \in W : \|W\| = 1\}$ is compact and T does not vanish on S_W^1, hence we have $C = \inf\{\|Tw\| : w \in S_W^1\} > 0$. We have

$$\frac{Pv}{\|Pv\|} \in S_W^1 \Rightarrow \left\| T\left(\frac{Pv}{\|Pv\|}\right) \right\| \geq C \Rightarrow \|Tv\| = \|T(Pv)\| \geq C \|Pv\|$$

Since $\varepsilon < d([v], \mathbf{P}(\ker(T))) = \frac{\|Pv\|}{\|v\|}$, and we can assume without loss of generality that $\|v\| = 1$ using $[v] = \left[\frac{v}{\|v\|}\right]$, we have $\|Pv\| > \varepsilon$, which implies $\|Tv\| > C\varepsilon$. □

Notation Let $A \subseteq \mathbf{P}(V)$. We will denote by $(A)_\varepsilon$ the ε-neighborhood of A in the metric defined above, i.e. $(A)_\varepsilon = \{x \in \mathbf{P}(V) | d(A, x) < \varepsilon\}$.

Definition 4.7. Let $\varepsilon \in (0, 1)$ and let $g \in \mathrm{PGL}(V)$.

- g is called (r, ε)-*proximal* $(r > 2\varepsilon)$ if there exists a point $a_g \in \mathbf{P}(V)$ and a hyperplane $H_g \subseteq \mathbf{P}(V)$ such that $d(H_g, a_g) > r$ and $g\left(\mathbf{P}(V) \setminus (H_g)_\varepsilon\right) \subseteq (a_g)_\varepsilon$.
- g is called (r, ε)-*very proximal* if both g and g^{-1} are (r, ε)-proximal.
- g is called *proximal* (resp. *very proximal*) if g is (r, ε)-proximal (resp. (r, ε)-very proximal) for some $r > 2\varepsilon > 0$.

Notation If g is very proximal we will denote $H^+ = H_g$, $H^- = H_{g^{-1}}$, $a^+ = a_g$ and $a^- = a_{g^{-1}}$

A nice property, which is of great importance for us, is the following result due to Breuillard and Gelander (see [BG1]):

Lemma 4.8 (Lemma 3.1 in [BG1])**.** *Let k be a valued field. Take $g \in \mathrm{PGL}_n(k)$ and let $\varepsilon \in \left(0, \frac{1}{4}\right)$. There exist two constants $c_1, c_2 \geq 1$ (depending only on the field k) such that if g is an (r, ε)-proximal transformation with $r \geq c_1\varepsilon$ then it must fix a unique point $\overline{v}_g$ inside its attracting neighborhood and a unique projective hyperplane $\overline{H}_g$ lying inside its repelling neighborhood. Moreover, if $r \geq c_1\varepsilon^{2/3}$, then all positive powers g^n, $n \geq 1$, are $\left(r - 2\varepsilon, (c_2\varepsilon)^{\frac{n}{3}}\right)$-proximal transformations with respect to these same $\overline{v}_g$ and $\overline{H}_g$.*

4.3 Furstenberg's Lemma

Definition 4.9. Let V be a vector space over a field F. A *linear sub-variety* in $\mathbf{P}(V)$ is $\mathbf{P}(W)$ where $W \leq V$ is a subspace. A *quasi-linear sub-variety* in $\mathbf{P}(V)$ is a finite union of linear sub-varieties.

Lemma 4.10 (Lemma 2 in [F])**.** *Let $T_n \in \mathrm{GL}(V)$ and let $\overline{T}_n$ denote the corresponding projective transformations. Assume $\det T_n = 1$ and $\|T_n\| \to \infty$ where $\|\cdot\|$ is a suitable norm on the linear endomorphisms of V. There exists a map π of $\mathbf{P}(V)$ whose range is a quasi-linear sub-variety $\subsetneq \mathbf{P}(V)$, and a sequence $\{n_k\}$ with $\overline{T}_{n_k}(x) \to \pi(x)$ for every $x \in \mathbf{P}(V)$.*

The transformation π of $\mathbf{P}(V)$ produced by the lemma is not a projectivization of a linear transformation of V, but it is not too far from such: we can find a linear transformation P of V such that $\overline{P} = \pi$ on $\mathbf{P}(V) \setminus \mathbf{P}(\ker(P))$. Using this, and assuming we already passed to a subsequence, we can say that $\overline{T}_n(x) \to \overline{P}(x)$ for every $x \in \mathbf{P}(V) \setminus \mathbf{P}(\ker(P))$.

Proposition 4.11. *For every $\varepsilon > 0$ there exists $N > 0$ such that for all $n > N$ we have*

$$\overline{T}_n\left(\mathbf{P}(V) \setminus (\mathbf{P}(\ker(P)))_\varepsilon\right) \subseteq (\mathbf{P}(\mathrm{Im}(P)))_\varepsilon$$

Proof. Applying Proposition 4.6.II) we have $C > 0$ such that if $d([v], \mathbf{P}(\ker(P))) > \varepsilon$ then $\|Pv\| > C\varepsilon$. Hence for all $[v] \in \mathbf{P}(V) \setminus (\mathbf{P}(\ker(P)))_\varepsilon$, $\|Pv\| > C\varepsilon$.
We will denote $S_n = P - T_n$. Then $\|S_n\| \to 0$ as $n \to \infty$. So

$$d([T_n v], \mathbf{P}(\mathrm{Im}(P)) = d([Pv - S_n v], \mathbf{P}(\mathrm{Im}(P)) \leq d([Pv - S_n v], [Pv])$$

Take N sufficiently large such that $\|S_n\|$ is small for all $n > N$. Since $\|Pv\| > C\varepsilon$, in particular it is far from 0, which implies that $d([Pv - S_n v], [Pv]) = \frac{\|S_n v \wedge Pv\|}{\|Pv - S_n v\| \|Pv\|}$ will be as small as we desire. Hence we can take N such that for all $n > N$ we will have $d([T_n v], [Pv]) < \varepsilon$, which implies the result. □

This property will be of great importance for us.

Example 4.12. Let $V = \mathbb{R}^2$. Define $T_n = \begin{pmatrix} \frac{1}{n} & \frac{1}{n^2} \\ n^2 & 2n \end{pmatrix}$. Then $\overline{T}_n = \overline{\begin{pmatrix} \frac{1}{n^3} & \frac{1}{n^4} \\ 1 & \frac{2}{n} \end{pmatrix}}$. Clearly T_n satisfies the conditions of the lemma (taking, for example, the norm to be $\|A\| = \max_{i,j} |a_{ij}|$). If we define $P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and denote $z = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$, then for all $x \in \mathbf{P}(V) \setminus \{z\}$ we have $\overline{T}_n(x) \to \overline{P}(x)$ as $n \to \infty$. So we can define $\pi = \overline{P}$ on $\mathbf{P}(V) \setminus \{z\}$. To define π on z, we re-write $\overline{T}_n$ as $\overline{T}_n = \overline{\begin{pmatrix} \frac{1}{n^2} & \frac{1}{n^3} \\ n & 2 \end{pmatrix}}$. Then we have $\overline{T}_n(z) \to \left[\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right] = z$ as $n \to \infty$. So we define $\pi(z) = z$, which completes the construction.

4.4 Attempts to generalize the proof from convergence groups to linear groups

Lemma 4.13. *Let V be a vector space over a field F and let $W, U \leq V$ be two subspaces. Then the set $\mathcal{C}(W,U) = \{T \in \mathrm{GL}(V) | T(W) \cap U \neq \{0\}\}$ is closed in the Zariski topology on $\mathrm{GL}(V)$.*

Proof. For a given $T \in \mathrm{GL}(V)$, observe that $T(W) \cap U \neq \{0\}$ is equivalent to $\exists w \in W \quad Tw \in U$. Fix any projection $p : V \to V$ s.t. $\mathrm{Im}(p) = U$. Hence the last condition could be written as: $\exists w \in W \ pTw = Tw$. Which could be reformulated as: $\ker(pT - T) \cap W \neq \{0\}$. It follows that $T \in \mathcal{C}(W,U) \Leftrightarrow$ the matrix representing $(pT - T)\restriction_W$, which is $\dim W \times \dim V$, does not have maximal rank $\Leftrightarrow \dim W \times \dim W$ minors in that matrix are zero, which is a polynomial condition on the coefficients of T. Hence $\mathcal{C}(W,U)$ is closed in the Zariski topology on $\mathrm{GL}(V)$. □

Corollary 4.14. *Let $\Gamma \leq \mathrm{GL}(V)$ be a Zariski-connected group, $W \leq V$ a subspace of co-dimension 1 and $u \in V$ such that $\Gamma u \nsubseteq W$. Then there exists $\gamma \in \Gamma$ such that $\gamma u, \gamma^{-1} u \notin W$.*

Proof. Set $U = \mathrm{Span}\langle u \rangle$. Let $\mathcal{C}(W,U)$ be the set defined in Lemma 4.13. It is closed, hence $\mathcal{D}_1 = \mathcal{D}(W,U) = \mathrm{GL}(V) \setminus \mathcal{C}(W,U)$ and $\mathcal{D}_2 = \mathcal{D}(U,W) = \mathrm{GL}(V) \setminus \mathcal{C}(U,W)$ are open.

- Since $\Gamma u \nsubseteq W$, there exists $\gamma \in \Gamma$ such that $\gamma u \notin W$. Hence $\gamma \in \mathcal{D}_2$. In particular, $\Gamma \cap \mathcal{D}_2 \neq \emptyset$.
- We have $\gamma u \notin W$, hence: $u \notin \gamma^{-1} W$. Thus $\gamma^{-1} \in \mathcal{D}_1$. In particular, $\Gamma \cap \mathcal{D}_1 \neq \emptyset$.

Now, $\Gamma \cap \mathcal{D}_1$ and $\Gamma \cap \mathcal{D}_2$ are non-empty open sets and Γ is connected (which is equivalent to irriducible. See Proposition 1.3.13 in [Ge]). Thus $(\Gamma \cap \mathcal{D}_1) \cap (\Gamma \cap \mathcal{D}_2) = \Gamma \cap (\mathcal{D}_1 \cap \mathcal{D}_2)$ is non-empty. Any $\gamma \in \Gamma$ such that $\gamma \in \Gamma \cap (\mathcal{D}_1 \cap \mathcal{D}_2)$ suits us. □

Remark Let Γ be a linear group equipped with the Zariski topology. We will denote by $\overline{\Gamma}^Z$ the Zariski closure of Γ and by $\left(\overline{\Gamma}^Z\right)^0$ the connected component of $\overline{\Gamma}^Z$ which contains 1. Define $\Gamma^0 = \Gamma \cap \left(\overline{\Gamma}^Z\right)^0$. Γ^0 is a normal subgroup of finite index in Γ (see Lemma 5.2 in [W]).

Before continuing treating the almost simple case, let us cite the following important result:

Definition 4.15. Let G be a group. A linear representation $\rho : G \to \mathrm{GL}(V)$ is said to be *strongly irreducible* if one of the following equivalent conditions holds:

- Let $H \leq G$ be a finite index subgroup of G. The restricted representation $\rho\restriction_H : H \to \mathrm{GL}(V)$ is irreducible.
- For any choice of propper subspaces $W_1, ..., W_m \subsetneq V$, $W_1 \cup ... \cup W_m$ is not a G-inavriant subset.

Theorem 4.16 (Theorem 7.6 and Lemma 7.7 from [GG])**.** *Let K be a field and $\mathbb{H}$ an algebraic K-group for which the connected component $\mathbb{H}^0$ is not solvable, and let $\Gamma < \mathbb{H}$ be a Zariski dense countable non-torsion subgroup. Then we can find a number $r > 0$, a valued field k, an embedding $K \hookrightarrow k$, an integer n, and a strongly irreducible projective representation $\rho : \mathbb{H}(k) \to PGL_n(k)$ defined over k, such that for any $\varepsilon \in \left(0, \frac{r}{2}\right)$ there is $g \in \Gamma \cap \mathbb{H}^0$ for which $\rho(g)$ acts as an (r, ε)-very proximal transformation on $\mathbf{P}(k^n)$.*

Lemma 4.17. *Let Γ be a sharply 2-transitive countable non-torsion linear group and assume Γ is primitive of the almost simple type. Then there exist $g \in \Gamma$ and $\sigma \in \mathrm{Inv}(\Gamma)$ such that $||g^{-n}\sigma g^n|| \to \infty$ as $n \to \infty$.*

Proof. Applying Theorem 4.16 on Γ and $\mathbb{H} = \overline{\rho(\Gamma)}^Z$, we get $r > 0$, a valued field k, a strongly irreducible projective representation $\rho : \Gamma \to PGL_n(k)$ and such that for any $\varepsilon \in \left(0, \frac{r}{2}\right)$ there is $g \in \Gamma \cap \mathbb{H}^0$ for which $\rho(g)$ acts as an (r, ε)-very proximal transformation on $\mathbf{P}(k^n)$. Using Lemma 4.8, we acquire the constants $c_1, c_2 \geq 1$. Since we have a lot of freedom in choice of ε, we can assume that $r \geq c_1\varepsilon^{2/3}$. Now fix g such that $\rho(g)$ is (r, ε)-very proximal. Denote the associated points and hyperplanes by a^+, a^-, H^+, H^-. It follows from the lemma that all positive powers g^n, $n \geq 1$, are $\left(r - 2\varepsilon, (c_2\varepsilon)^{\frac{n}{3}}\right)$-proximal with respect to these same a^+ and H^+. The same is is true for g^{-1}, since it is also (r, ε)-proximal.
Observe that any non-trivial involution has exactly two eigenvalues: $1, -1$. Take any involution $\tau \in \mathrm{Inv}(\Gamma)$. Let $[v], [w]$ be two eigenvectors of $\rho(\tau)$ associated to $1, -1$ respectively. From the strongly irreducibility of the representation, there exist $\gamma_1, \gamma_2 \in \Gamma^0$, since Γ^0 is a finite index subgroup, such that $\rho(\gamma_1)[v] \notin H^-$ and $\rho(\gamma_2)[w] \notin H^-$. Using the same argument as in Corollary 4.14 on Γ^0, we get an element $\gamma \in \Gamma^0 \leq \Gamma$ such that $\rho(\gamma)[v] \notin H^-$ and $\rho(\gamma)[w] \notin H^-$. Define $\sigma = \gamma\tau\gamma^{-1}$. σ is an involution and $\rho(\gamma)[v], \rho(\gamma)[w]$ are it's eigenvectors associated to $1, -1$ respectively. From the construction, they are both outside H^-. Denote $\sigma_n = g^{-n}\sigma g^n$.
We want to prove that $\{||\sigma_n||\}_{n=0}^{\infty}$ is unbounded. Assume it is bounded, then $\{\sigma_n\}_{n=0}^{\infty}$ is contained in some compact set $K \subseteq \Gamma$. Define $f : K \to [0, 1]$ by

$$f(x) = \max\left\{ \frac{|< v, w >|}{\|v\|\,\|w\|} \,\middle|\, v, w \text{ are eigenvectors of } x \text{ associated to different eigenvalues} \right\}$$

i.e. f gives the maximal cosine of an angle between different eigenspaces. Observe that $f(x) < 1$ for all $x \in K$ since two eigenvectors associated to different eigenvalues are linearly independent, hence the angle between them can not be 0. f is continuous and K compact, so f achieves a maximum $m < 1$.
On the other hand, observe that if $\sigma v = \lambda v$ then $\sigma_n(g^{-n}v) = \lambda g^{-n}v$. From the selection of σ we have two eigenvectors v, w such that $[v], [w] \notin H^-$. Since $g^{-n} = \left(g^{-1}\right)^n$ is $\left(r - 2\varepsilon, (c_2\varepsilon)^{\frac{n}{3}}\right)$-proximal with respect to these same a^- and H^-, we have $\rho(g^{-n})[v], \rho(g^{-n})[w] \in (a^-)_{(c_2\varepsilon)^{\frac{n}{3}}}$. Since as $n \to \infty$ we have $(c_2\varepsilon)^{\frac{n}{3}} \to 0$, we can find eigenvectors of σ_n as close as we wish, hence $f(\sigma_n) \to 1$ as $n \to \infty$. A contradiction. □

Remark When a projective representation is given, we will denote $\rho(\gamma)$ by $\overline{\gamma}$ for all $\gamma \in \Gamma$.

Notation Let Γ be a sharply 2-transitive countable non-torsion linear group and assume Γ is primitive of the almost simple type. Fix $g \in \Gamma$, $\sigma \in \mathrm{Inv}(\Gamma)$ and the strongly irreducible projective representation $\rho : \Gamma \to PGL_n(k)$ given by Lemma 4.17. Define $\sigma_n = g^{-n}\sigma g^n$. Then $||\sigma_n|| \to \infty$ as $n \to \infty$.
By Furstenberg's Lemma (4.10), there exists a subsequence σ_{n_k} such that $\overline{\sigma_{n_k}}(x) \to \pi(x)$ for all $x \in \mathbf{P}(k^n)$, where π is a singular transformation on $P(k^n)$. Denote by P the linear transformation of V such that $\overline{P} = \pi$ on $\mathbf{P}(V) \setminus \mathbf{P}(\ker(P))$, as described after Lemma 4.10. Define $\mathrm{N} = \ker(P)$ and $\mathrm{R} = \mathrm{Im}(P)$.

Theorem 4.18. *Let Γ be a sharply 2-transitive countable non-torsion linear group and assume Γ is primitive of the almost simple type. Let* N, R *be the sets defined in the previous Notation. Assume further that there exists $\gamma \in \Gamma$ such that $\overline{\gamma}(\mathbf{P}(\mathrm{N})) \cap \mathbf{P}(\mathrm{R}) = \overline{\gamma}(\mathbf{P}(\mathrm{R})) \cap \mathbf{P}(\mathrm{N}) = \emptyset$. Then p-char$(\Gamma) = 0$.*

Proof. Continuing from the Notation, for all $\varepsilon > 0$ there exists N such that for all $n > N$ and for all $x \notin (\mathbf{P}(\mathrm{N}))_\varepsilon$, $\overline{\sigma_n}(x) \in (\mathbf{P}(\mathrm{R}))_\varepsilon$, upon passing to a subsequence.
Fix the γ that is given by the assumption on Γ.
Define $\tau_n = \gamma^{-1}\sigma_n\gamma$. Now $\overline{\tau}_n$ converges to $\phi = \overline{\gamma}^{-1}\pi\overline{\gamma}$. Denote by P' the corresponding linear transformation of V, as before. Define $\mathrm{N}' = \ker(P')$ and $\mathrm{R}' = \mathrm{Im}(P')$.
By the assumption on γ, $\mathbf{P}(\mathrm{R}') = \overline{\gamma}^{-1}(\mathbf{P}(\mathrm{R}))$ is disjoint from $\mathbf{P}(\mathrm{N})$ and $\mathbf{P}(\mathrm{N}') = \overline{\gamma}^{-1}(\mathbf{P}(\mathrm{N}))$ is disjoint from $\mathbf{P}(\mathrm{R})$. Now, take $\varepsilon > 0$ such that $(\mathbf{P}(\mathrm{R}'))_\varepsilon \cap (\mathbf{P}(\mathrm{N}))_\varepsilon = \emptyset$ and $(\mathbf{P}(\mathrm{R}))_\varepsilon \cap (\mathbf{P}(\mathrm{N}'))_\varepsilon = \emptyset$. By the construction, for some n large enough we have:

$$\overline{\sigma}_n \left(\mathbf{P}(k^n) \setminus (\mathbf{P}(\mathrm{N}))_\varepsilon\right) \subseteq (\mathbf{P}(\mathrm{R}))_\varepsilon, \quad \overline{\tau}_n \left(\mathbf{P}(k^n) \setminus (\mathbf{P}(\mathrm{N}'))_\varepsilon\right) \subseteq (\mathbf{P}(\mathrm{R}'))_\varepsilon$$

Now take $x \in P\left(\wedge^k V\right) \setminus ((\mathbf{P}(\mathrm{N}))_\varepsilon \cup (\mathbf{P}(\mathrm{R}))_\varepsilon \cup (\mathbf{P}(\mathrm{R}'))_\varepsilon)$. Observe $\overline{(\tau_n\sigma_n)^m}(x)$ for all $m > 0$: $x \notin (\mathbf{P}(\mathrm{N}))_\varepsilon$, hence $\overline{\sigma_n}(x) \in (\mathbf{P}(\mathrm{R}))_\varepsilon$. Since $(\mathbf{P}(\mathrm{R}))_\varepsilon \cap (\mathbf{P}(\mathrm{N}'))_\varepsilon = \emptyset$, $\overline{\sigma_n}(x) \notin (\mathbf{P}(\mathrm{N}'))_\varepsilon$ and hence $\overline{\tau_n\sigma_n}(x) \in (\mathbf{P}(\mathrm{R}'))_\varepsilon$. Repeating the same argument inductively, we get that for all $m > 0$, $\overline{(\tau_n\sigma_n)^m}(x) \in (\mathbf{P}(\mathrm{R}'))_\varepsilon$. In particular, $\overline{(\tau_n\sigma_n)^m}(x) \neq x$. Hence $\tau_n\sigma_n$ is of infinite order, and by the definition, p-char$(\Gamma) = 0$. □

Remark We believe that the additional assumption on Γ, i.e. the existence of the specific element, is fulfilled for every such Γ.

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