Stationary Dynamical Systems and the Furstenberg-Poisson boundary

Graduate course, Northwestern University

Yair Hartman

Last Update: Wednesday 14th March, 2018

1. Introduction

Lecture 1

Stationary theory is a a theory that was developed by Furstenberg in the 60's and the 70's which includes the study of the Furstenberg-Poisson boundary.

It is a branch of ergodic theory that deals with measurable groups action which has several sides. It is closely related to random walks on groups. Hence, different people find it interesting for different reasons.

Before starting from the very basics, explaining and defining every object, I want to spend sometime over viewing the theory. Hence, if we mention terms that you are not familiar with, that's fine - we will define everything when we will get there. So now just sit back and enjoy the view!

In general I want to present 3 perspectives.

TODO: discuss first the probability approach

Group theory - Application to rigidity theory

I'm mostly interested in this theory as a tool to study groups. This theory is especially relevant and useful for the study of large infinite groups. Most of it is not relevant to abelian, or nilpotent groups. And while some of it relevant to solvable groups, the main focus is on non-amenable groups.

Here is a example:

Theorem 1.0.1 — Margulis' normal subgroup theorem 67'. Let G be a connected simple Lie group with finite center and $rk_{\mathbb{R}} \geq 2$ and let Γ be an irreducible lattice. Then any normal subgroup N in Γ is either finite or is of finite index.

The first example of such Γ is $SL_3(\mathbb{Z})$. This theorem classifies the normal subgroups of $SL_3(\mathbb{Z})$ and a-priory seems to be totally unrelated to Poisson boundary and random walks. In some way, the abstract group $SL_3(\mathbb{Z})$ remembers that it came from the group $SL_3(\mathbb{R})$. A natural question is in what sense are they similar.

In what sense $\mathbb{Z}$ or $\mathbb{Z}^n$ are similar to $\mathbb{R}$ or $\mathbb{R}^3$. The lattices are countable discrete groups and the ambient groups are connected - so they seem very different. Yet, they are similar in a way. The basic idea is that if we look at $\mathbb{Z}$ from very far away, we might think that we are looking at $\mathbb{R}$.

But, "far away" in what sense? One approach, is to define far away using a random walk. For that, one fixes a some random walk on $\mathbb{Z}^3$, or $SL_3(\mathbb{Z})$ say, and then far a way would mean after a long time. Indeed, in some way, these processes will look like random walks on $\mathbb{R}^3$ and $SL_3(\mathbb{R})$.

A stronger application, is the following.

Theorem 1.0.2 — Stuck-Zimmer theorem 94'. Let G as before. Then any faithful measure preserving action of G is either transitive, or essentially free.

This theorem classifies the stabilizers of measure preserving actions of such groups, which can be thought of as a generalization of normal subgroups, and indeed, this theorem implies the normal subgroup theorem.

Again, while this result deals with measure preserving actions, the main tool here is a structure theorem of certain stationary actions.

These two results, are stated here in the context of Lie groups, but in some sense, the main machinery in these proofs relay of abstract stationary theory more than working with the Lie group structure. And indeed, these results were proved by Bader-Shalom ('06) in a more abstract setup. In this course we will prove these results.

Probability - Random walks on groups

Loosely speaking, the Furstenberg-Poisson boundary is an objects that captures the asymptotic behavior of the random walk. Although it is a measurable space, one can think of it as sort of compactification of the group that is constructed out of a random walk. In the discussion above, about the relation between lattices and their ambient groups, we said that they are similar in the sense that they share the same compactification.

There are many questions in probability that are related to random walks on groups, and these are very active research fields. We will prove the existence of the Furstenberg-Poisson boundary, for every random walk on every group. In fact, the whole theory works perfectly fine for locally compact second countable groups. However, we will develop the theory in the context of discrete countable groups, for simplicity.

In any case, a lot of research is concern with concrete realizations of the Furstenberg-Poisson boundary as measures on natural topological spaces that the group is question is acting on. For many classes of groups there are natural boundaries: Gromov boundary of hyperbolic groups, flag varieties of linear groups, Thurston's boundary of mapping class groups, and so on. In all these case, using an entropy theory that we will develop, one can realize the Furstenberg-Poisson boundary on these topological boundaries.

Theorem 1.0.3 Consider the simple random walk on the free group $\mathbb{F}_2$. Show that the boundary of the tree, has a measure which is the Furstenberg-Poisson boundary.

Ergodic Theory - A good generalization of measure preserving actions

The main focus of ergodic theory is the study of measure preserving actions. For convenience, we can think of an action on a compact space where the measure doesn't move when we hit it with group's elements.

The theory behind these kind of actions is quite developed. For example, the celebrated ergodic theorem holds:

Theorem 1.0.4 — Birkhoff. Let (X,m,T) be an ergodic measure preserving system, and let

$f \in L^1(X,m)$. The for m-almost every x,

$$\lim_n \sum_{k=0}^{n} f(T^k(x)) = \int_X f(x)dm(x)$$

When T as an invertible transformation, so we are actually dealing with a $\mathbb{Z}$-action. A natural generalization holds for amenable groups:

Theorem 1.0.5 Let G be a discrete countable group, $G \curvearrowright (X,m)$ be an ergodic measure preserving action, $f \in L^1(X,m)$ and F_n be a tempered Folner sequence. Then m-almost every x,

$$\lim_n \frac{1}{|F_n|} \sum_{g\in F_n} f(gx) = \int_X f(x)dm(x)$$

The existence of Folner sequence allows to take averages. So in the classical setup, there are two ingridients: a natural averaging scheme and the measure, being invariant.

1. Can we say anything about measure preserving actions of non amenable groups?
2. Can we do anything with measures which are not invariant?

These two questions are related in some sense: whenever an amenable groups acts on a compact space, there is always invariant measure. Hence leaving the world of amenable groups, both questions are relevant: sometimes we will have invariant measures, but we won't know how to take averages, and sometimes, there will be no invariant measures at all and in that case, we might want to consider something weaker that invariant measure.

The nice feature of invariant measures, is that they are invariant. When considering measures which are not invariant, they changes as we apply sequence of group's elements. Consider for example the situation of the ergodic theorem. The fact that the measure is invariant, means that when we hit it with these growing Folner sequences, we see different points, be we always see the same measure. If the measure is not invariant, then things are starting to move and we loose control.

Here again the idea of a random walk becomes useful, addressing both issues. Given a random walk on a group we will consider measures which are stationary, with respect to this random walk. These will be the generalization of invariant measures that are always exist. At the same time, having a random walk, gives a natural averaging schemes, and indeed there is a nice ergodic theorem in this context, called *Kakutani's random ergodic theorem*.

Some terms

Groups

1. nilpotent groups
2. amenable groups
3. Cayley graph
4. growth rate of a group
5. the free group
6. lamplighter groups, affine groups

Group actions

1. measurable actions
2. ergodic measures, ergodic theorem
3. non-singular actions
4. factors
5. universal objects

Analysis

1. L^1, L^2, L^∞ spaces
2. Radon-Nikodym
3. Gelfand's theory

Probability

1. Random walks
2. Harmonic functions
3. Marginals
4. Shannon entropy

2. Background on measurable actions

Lecture 2

2.1 Measurable and topological actions

2.1.1 Borel spaces

Definition 1
1. A **Borel space** is a pair $(X, \mathscr{B})$ where X is a set and $\mathscr{B}$ is a σ-algebra on X, that is, a collection of subsets of X which contains $\emptyset$ and X and is closed under taking complements and countable unions. The elements $E \in \mathscr{B}$ are called **Borel sets** or measurable sets.
2. If $(X_1, \mathscr{B}_1)$ and $(X_2, \mathscr{B}_2)$ are Borel spaces, a map $\varphi : X_1 \to X_2$ is a **Borel map** (or measurable) if $\varphi^{-1}(B) \in \mathscr{B}_1$ for all $B \in \mathscr{B}_2$.
3. φ is an **isomorphism** if it is a bijection between X_1 and X_2 and its inverse is also measurable.
4. $(Y, \mathscr{A})$ is a **Borel subspace** of $(X, \mathscr{B})$ if $Y \subset X$ and there exists some $B \in \mathscr{B}$ such that $\mathscr{A} = \mathscr{B} \cap B$.

■ **Example 2.1** Let X be a topological space, and consider $\mathscr{B}$ to be the σ-algebra generated by the open subsets.

Definition 2 $(X, \mathscr{B})$ is **a standard Borel space** if it is isomorphic to a Borel subspace of a complete metric space (in particular, it is separable).

All the spaces that we will meet along the way will be standard Borel spaces. But before that, let's say some things on the level of Borel spaces.

Facts 1
- The cardinality of a standard Borel space is either finite, countable or continuum.
- Two standard Borel spaces are isomorphic if and only if they have the same cardinality. It says that is a sense, there is not enough structure in a Borel space. Therefore, we will add another ingredient to the story - measures.
- In particular, any uncountable standard Borel space is isomorphic to the unit interval.
- Most importantly: Any standard Borel space is **separable**, meaning there there exists a countable collection of measurable sets B_n such that $\mathscr{B} = \sigma(\{B_n\})$ and for any $x \neq y$,

there exists n such that $x \in B_n$ and $y \notin B_n$).

2.1.2 Borel G-spaces

From now on, G will always stand for a discrete countable group. The theory that we will develop holds to locally compact second countable groups, such as Lie groups, without to much adaptations. Since it is rich and interesting already for discrete countable groups, for simplicity, we won't present this generality here.

Given a Borel space $(X, \mathscr{B})$, Let $\mathrm{Aut}(X, \mathscr{B})$ denote the group of all Borel-automorphisms of X.

Definition 3 X is a **Borel G-space**, if $G \curvearrowright (X, \mathscr{B})$ in a measurable way, that is:

There exists an *action map* $G \times X \xrightarrow{a} X$ which is measurable. We will simply write $gx = a(g, x)$, so any g is an isomorphism $X \to X$. The map a is an action in the sense that $g_1 g_2 x = g_1(g_2 x)$.

Another way do describe it is by a group homomorphism : $G \to \mathrm{Aut}(X, \mathscr{B})$.

We will also consider topological actions. All of the topological spaces are assumed to be Hausdorff, and usually we consider compact spaces.

Definition 4 A compact space X is a **G-space**, if $G \curvearrowright X$ continuously, that is:

There exists an *action map* $G \times X \xrightarrow{a} X$ which is continuous.

If $\mathrm{Homeo}(X)$ denotes the group of all homeomorphisms from X to itself, then a continuous action is a group homomorphism $G \to \mathrm{Homeo}(X)$.

Remark 1 The view of an action as a group homomorphism is useful to define these action for more general groups.

Since every continuous function is Borel (as usual, when we already have a topology, the Borel structure is the one generated by the topology), we get that $\mathrm{Homeo}(X)$ is a subgroup of $\mathrm{Aut}(X, \mathscr{B})$. In particular, any continuous action is a Borel action.

Let $(X, \mathscr{B})$ and $(Y, \mathscr{A})$ be two G-spaces. We say that a Borel map $\pi : X \to Y$ is a **G-map** if it is equivariant, namely $\pi(gx) = g\pi(x)$ for every $x \in X$ and $g \in G$.

We will usually require maps between compact spaces to be continuous.

Definition 5 Let $(X, \mathscr{B})$ be a Borel G-space. A compact metric G-space Y is a **compact model** of X if they are G-isomorphic as measurable spaces.

Remark 2 The unit interval is a compact model of every uncountable Borel space in the level of space, but not as G-spaces. Given an abstract Borel G-space we get an action on the unit interval, but this action is only Borel, and not continuous in general.

Theorem 1 — The compact model theorem. Any standard Borel G-space admits a compact model.

If the Borel space is not standard, then one can still find compact models, but there is no guaranteed that this model would be metrizable.

In some way, a topological model is similar to providing coordinate system to an abstract space. A given Borel space admits many different compact models. However, sometimes there are properties of the abstract space that are naturally reflected in a topological realizations. We will see an example of such a property of abstract space, that is equivalent to a some topological property, for any compact model.

One important corollary of the compact model theorem is that for any a Borel G-space, the stabilizer subgroups, $\mathrm{Stab}(x) = \{g | gx = x\}$ are closed subgroup of G (although it is not relevant for our setup of discrete groups).

2.2 Measured Borel Spaces

Definition 6 A **standard measured space** is a triple $(X,\mathscr{B},\nu)$ where

- $(X,\mathscr{B})$ is a standard Borel space.
- $\nu:\mathscr{B}\to[0,\infty]$ is a measure, namely, it is σ-additive linear function: if $\{E_n\}\subset\mathscr{F}$ is pairwise disjoint then $\nu\left(\bigcup E_n\right)=\sum\nu\left(E_n\right)$.
  We will mainly focus on probability measures, that is ν such that $\nu(X)=1$.

Once we have a measure in the picture, we want to ignore things that the measure doesn't see. Measurable function on a measured space are only defined up to null sets. Hence functions are not really functions on the set but only equivalence class of functions. Similarly, if (X,ν) and (Y,η) are two measured spaces, then when we say that $\pi:X\to Y$ is a measurable map, we mean that there exists $X'\subset X$ and $Y'\subset Y$, with $\nu(X')=\eta(Y')=1$ and the map $\pi:X'\to Y'$ is a Borel isomorphism.

On one hand, it becomes a bit harder to work in this category - there are no points in the space, but only sets. No functions, only equivalence class of functions. On the other hand, the exact space abstractness makes this category very flexible. We will talk soon about factors and will see things that are possible and very useful, that we cannot get in the topological setup.

Let $\pi:X\to X$ be a Borel map. Given a measure ν on X, the **push-forward measure** $\pi_*\nu$ is the measure on Y defined by $\pi_*\nu(E)=\nu(\pi^{-1}(E))$.

Two measures ν_1,ν_2 on the the same Borel space X are said to be equivalent, or in the same **measure-class** if $\nu_1(E)=0\iff\nu_2(E)=0$. In that case both ν_1 is absolutely continuous with respect to ν_2 and also in the other direction and we denoted it by $\nu_1\sim\nu_2$.

Given a measured Borel space, let $\mathrm{Aut}^*(X,\nu)$ denote the group all the isomorphism $\phi:X\to$ such that $\nu\sim\phi_*\nu$. So $\mathrm{Aut}^*(X,\nu)\le\mathrm{Aut}(X,\mathscr{B})$.

We will only consider **non-singular actions** on standard measure spaces, meaning that $g_*\nu=g\nu$ and ν are equivalent for any g. Such measures are called **quasi-invariant measures**. In other words, the action map is a measurable group homomorphism $G\to\mathrm{Aut}^*(X,\nu)$.

Finally, an action said to be **a measure preserving action** if $\nu=g\nu$ for all g. In that case we say that ν is an **invariant measure**.

Again, if $\mathrm{Aut}(X,\nu)$ denote the subgroup of $\mathrm{Aut}^*(X,\nu)$ of isomorphisms $\phi:X\to X$ such that $\phi_*\nu=\nu$ then a measure preserving action is just such where the image of the action map is in $\mathrm{Aut}(X,\nu)$.

The space of probabilities

We now mix topological and measurable actions.

Let X be a compact space. We denote by $\mathscr{C}(X)$ the algebra of all continuous functions $X\to\mathbb{C}$. Riesz representation theorem states that measures on X (that are defined on the Borel sigma-algebra) are the same as positive linear functional on $\mathscr{C}(X)$. In particular, the space of all probability measures on X, that we denote by $\mathrm{Prob}(X)$ is a subset of the dual space $\mathscr{C}(X)$.

The space $\mathrm{Prob}(X)$ is naturally equipped with the weak*-topology, namely $\nu_n\to\nu$ if for any $f\in\mathscr{C}(X)$, we have that $\nu_n(f)\to\nu(f)$. When X is compact we get that $\mathrm{Prob}(X)$ is a compact convex space.

When $G\curvearrowright X$ is a compact G-space, then we get a continuous action $G\curvearrowright\mathrm{Prob}(X)$, via the push-forwarding the measures. Fixed points in this action are exactly invariant measures.

Lecture 3

Ergodicity

Definition 7 A non-singular action $G\curvearrowright(X,\nu)$ is said to be **ergodic** if there are no non-trivial invariant subsets, that is if $gA=A$ for all $g\in G$ then $\nu(A)\in\{0,1\}$.

Exercise 2.1 Prove that m is ergodic if and only if any invariant function $f : X \to \mathbb{R}$ (that is, $f(x) = f(gx)$ for all $g \in G$) is constant.

Solution ■

Let X be a compact G-space. We denote by $\mathrm{Prob}_G(X) \subset \mathrm{Prob}(X)$ to be the subset of invariant measures on X. In general, it might be empty. However, $\mathrm{Prob}_G(X)$ is a convex compact space as well.

A point ν in a convex space is **extremal** if the only solution to $\nu = c\nu_1 + (1-c)\nu_2$ is when c is either 0 or 1.

Lemma 1 Let X be a compact G-space. The following are equivalent for an invariant measure $m \in \mathrm{Prob}_G(X)$:

1. m is ergodic, that is, if A is invariant ($gA = A$ for all $g \in G$), then $m(A) \in \{0,1\}$.
2. $m(A) \in \{0,1\}$ for all m-invariant A ($m(gA\Delta A)) = 0$ for all $g \in G$).
3. m is extremal in $\mathrm{Prob}_G(X)$.

Proof. ($1 \implies 2$) Clear.

($2 \implies 3$) Assume that m is not extremal and $m = cm_1 + (1-c)m_2$ where $0 < c < 1$. Then both m_1 and m_2 are absolutely continuous w.r.t m. Let $f_1(x) = \frac{dm_1}{dm}(x)$ be the Radon-Nikodym derivative. f_1 is the unique function that satisfies $\int_B f(x)\,dm = m_1(A)$. Since $m \neq m_1$, f_1 is not constant w.r.t. m, and so we can find $a > 0$ with $A = \{x | f(x) < a\}$ such that $0 > m(A) > 1$.

Since both m and m_1 are invariant, f_1 is G-invariant w.r.t. m. In particular, $m(gA\Delta A) = 0$ for all $g \in G$.

($3 \implies 1$) Assume that there is an invariant set A with $m(A) = c$ for $0 < c < 1$. Let $m_1(B) = \frac{1}{c}m(B\cap A)$ and $m_2(B) = \frac{1}{1-c}m(B \cap (X\setminus A))$, for any $B \in \mathscr{B}$. Note that m_1, m_2 are invariant measures.

Then

$$\begin{aligned}
\left(cm_1 + (1-c)m_2\right)(B) &= m(B\cap A) + m\left(B\cap(X\setminus A)\right) \\
&= m(B)
\end{aligned}$$

and so m is not extremal. ■

The fact that extremal measures are ergodic is very general. For the other direction we used the fact that if an invariant measure is absolutely continuous w.r.t. another invariant measure, then the Radon-Nikodym derivative is invariant function.

2.3 Factors

Definition 8 Let (X,ν) and (Y,η) be two measured G-spaces.

We say that (X,ν) is an **extension** of (Y,η), and (Y,η) is a **factor** of (X,ν) if there exists Borel G-map (**factor map**) $\pi : X \to Y$ such that $\pi_*\nu = \eta$.

When X and Y are compact spaces, we require the continuous equivariant map π to be onto.

There are several ways to think of a factor, we list here the ones that we will use. First a picture:

TODO: Draw a square over a line

Algebras Embeddings

A factor map π induces a natural embedding of algebras, via the "pull-back" map $\pi^* f = f(\pi(x))$.

In the topological category, we get an embedding of the algebra $\mathscr{C}(Y) \subset \mathscr{C}(X)$. Functions in $\mathscr{C}(X)$ that are coming from Y are constant on the fibers.

A factor in the measurable category $\pi : (X,\nu) \to (Y,\eta)$ gives a natural embedding of $L^\infty(Y,\eta) \subset L^\infty(X,\nu)$.

So factors yields to G-invariant closed sub-algebra. In fact, the other direction is also true. If we find say in $\mathscr{C}(X)$ a G-invariant closed sub-algebra, then it must be coming from a G-equivariant factor.

2.3.1 Measurable factors

As sub-sigma-algebras

Let $(X,\mathscr{B})$ be a Borel space and let ν a measure on X. A sub-sigma-algebra $\mathscr{A}$ is called **ν-complete** if any subset of a null set, $A \subset N$ where $\nu(N) = 0$, belongs to $A \in \mathscr{A}$. It is not a heavy requirement - given a measure ν, one can always complete any sigma-algebra by adding to it all the measurable subsets of the null sets.

Let $(X,\mathscr{B},\nu) \xrightarrow{\pi} (Y,\mathscr{A},\eta)$ be a factor map of measurable spaces. Consider the following sigma algebra on X consists of all the sets of the form $\mathscr{B}_Y = \left\{\pi^{-1}(E)\right\}_{E\in\mathscr{A}}$. In other words, we are pulling back the sigma-algebra from the factor.

By that we get a G-invariant sub-sigma algebra $\mathscr{B}_Y \subset \mathscr{B}$ which is complete. Note that it is invariant as a collection of subsets, not every set is invariant!

Theorem 2 — Macky's point realization. Let $(X,\mathscr{B},\nu)$ be a standard Borel space and let $\mathscr{A}' \subset \mathscr{B}$ be a sub-sigma algebra which is ν-complete. Then there exists a standard space $(Y,\mathscr{A},\eta)$ such that $(X,\mathscr{B},\nu) \xrightarrow{\pi} (Y,\mathscr{A},\eta)$ and $\mathscr{A}' = \pi^*\mathscr{A}$.

Now assume that $(X,\mathscr{B},\nu)$ is a G-space (non-singular action). The space Y admits a natural G-action (and the map is equivariant) if and only if $\mathscr{A}'$ is a G-invariant sigma-algebra.

Given $(X,\mathscr{B},\nu)$, we can consider the space $L^\infty(X,\mathscr{B},\nu)$. There is a natural correspondence between sub-sigma algebras $\mathscr{A} \subset \mathscr{B}$ and sub spaces $L^\infty(X,\mathscr{A},\nu) \subset L^\infty(X,\mathscr{B},\nu)$. The embedding is given by the sub space of $\mathscr{A}$-measurable functions. A $\mathscr{B}$-measurable function is $\mathscr{A}$-measurable if and only if it is of the form $\pi^* f$ for a function f on the Mackey realization of $\mathscr{A}$.

So, we can either work algebras or with sigma-algebras, and it is really matter of taste.

■ **Example 2.2** Let $G \curvearrowright (X,\mathscr{B},\nu)$, and let $K \le G$. Consider

$$\mathscr{B}_K = \left\{E \in \mathscr{B} | kE = E \forall k \in K\right\}.$$

Then $\mathscr{B}_K$ is a sub-sigma algebra and we denote by $K\backslash\backslash(X,\nu)$ the Mackey realization of $\mathscr{B}_K$. We equipped $K\backslash\backslash(X,\nu)$ with the pushed-forward ν measure on the Mackey space of $\mathscr{B}_K$.

We now note that for a normal subgroup $N \triangleleft G$, $N\backslash\backslash(X,\nu)$ is a G-space. Indeed, if E is N invariant then gE is also: $ngE = gg^{-1}ngE = gn'E = gE$.

Theorem 2.3.1 — Disintegration. Let X be a compact space, $\nu \in \mathrm{Prob}(X)$, and $(X,\nu) \xrightarrow{\pi} (Y,\eta)$ be a measurable factor map. Then there exist a Borel map $Y \to \mathrm{Prob}(X)$, $y \mapsto \nu_y$ such that

1. [Fibered measures] For ν-almost every $y \in Y$, $\nu_y\left(\pi^{-1}(y)\right) = 1$.

2. [Disintegration]

$$\nu = \int_Y \nu_y d\eta(y)$$

That is, for any Borel map $f : X \to \mathbb{C}$,

$$\int_X f(x)d\nu(x) = \int_Y \left(\int_X f(x)d\nu_y(x) \right) d\eta(y).$$

3. [Uniqueness] If there is another map which sends $y \mapsto \nu'_y$ then $\nu'_y = \nu_y$ for η-almost every y.

Example 2.3 — Ergodic decomposition. The space $Y = G\backslash\backslash(X,\nu)$ called the **space of G-ergodic components**. Consider $\pi : X \to Y$ and denote by η the pushed forward measure on Y.

First note that if ν is an ergodic measure then Y is the trivial action. There might be invariant measures, but all of them get ν-measure which is 0 or 1. It follows that Y is the trivial, one point space, with Dirac measure.

We claim that in general, ν_y is an ergodic measure for η-a.e. y. First, note that G acts trivially on $G\backslash\backslash X$ - since each set in the sigma algebra is G-invariant. It follows that almost every fiber $\pi^{-1}y$ is G-invariant set in X. Indeed, if $\pi(x) = y$ then $\pi(gx) = g\pi(x) = gy = y$.

Now let $E \subset X$ be G-invariant. By definition, it is already measurable in $Y = G\backslash\backslash X$, and we can consider $\pi(E) \subset Y$. If $y \in \pi(E)$ then $\nu_y(E) = 1$ and otherwise, $\nu_y(E) = 0$.

Remark 3 Now assume that the measure ν is an invariant measure. Since the measures ν_y are all ergodic, this decomposition presents ν as an integral extremal measures.

In finite dimensional convex spaces, it is clear that any point can be written as a convex combination of the extremal points. This holds also for infinite dimensional and it calls *Krein–Milman theorem*. So once we know that ergodic measures are extremal, then we can get that one can present any measure as an integral of ergodic.

Moreover, as this decomposition is unique, we get that $\mathrm{Prob}_G(X)$ is a *Choquet simplex* (which means that any point is being represented in a unique way as integral of the extremal).

Lecture 4

Disintegration and conditional expectations

As we saw, we can always pull back functions from a factor to its extension. However, the disintegration theorem implies that in the measurable category, one can also push forward functions from the extension to the factor.

Theorem 3 — Conditional expectation. Let $(X,\mathscr{B},\nu)$ and let $\mathscr{A} \subset \mathscr{B}$ be a sub-sigma algebra.

For a function $f \in L^1(X,\mathscr{B},\nu)$, there exists a unique function $\mathbb{E}(f|\mathscr{B}) \in L^1(X,\mathscr{A},\nu)$, such that for any set $A \in \mathscr{A}$,

$$\int_A f(x)\,d\nu(x) = \int_A \mathbb{E}(f|\mathscr{B})(x)\,d\nu(x).$$

Consider Y to be the Mackey space of $\mathscr{A}$. Then we can think of $\mathbb{E}(f|\mathscr{B}) = \pi_* f$ as a function on Y. How the function $\pi_* f$ decides what to give to y? There are many values in the fiber $\pi^{-1}(y)$. So the conditional expectation integrates the fiber according to ν_y.

Note that for any function f on Y we get $\pi_*(\pi^* f) = f$, but not the other way around! In fact, $\pi^*(\pi_* f) = \mathbb{E}(f|\mathscr{A})$ where $\mathscr{A}$ is the pull back of the σ-algebra from the factor.

Relative compact model

If we have a measurable factor $\pi : (X, \nu) \to (Y, \eta)$ we can build a compact model for each of them. So we get two compact G-spaces, $\bar{X}$ and $\bar{Y}$. Also, we get a measurable factor map $\varphi : (X, \nu) \to (Y, \eta)$ such that $\varphi = \pi$, ν-almost everywhere. But, we cannot guaranteed φ to be continuous with respect to the topologies that we got. One may hope to replace it by another factor map that will be continuous. However, such a map do not exists in general. The best we can get are two compact spaces with a measurable map between them.

Exercise 2.2 The properties non-singular, invariant and ergodic are preserved under factors.

Add

Why we can push measures forward?

Why considering measures is natural, even if the space is a topological space?

2.3.2 Martingales

Let us recall a fundamental theorem from probability theory.

$\mathscr{B}_n$ is increasing σ-algebras (called filtration) in some probability space $(X, \mathscr{B}, \nu)$, that **tends** to $\mathscr{B}$. Meaning that $\mathscr{B}$ is the sigma algebra generated by the sets in $\bigcup_n \mathscr{B}_n$.

Definition 9 A sequence of random variables $\{M_n\}$ is **a bounded martingale w.r.t.** $\{\mathscr{B}_n\}$ if

- $\mathbb{E}(|M_n|) < \infty$
- M_n is $\mathscr{B}_n$-measurable
- $\mathbb{E}(M_{n+1}|\mathscr{B}_n) = M_n$

If we realize each sigma algebra $\mathscr{B}_n$ as a space (X_n, ν_n) so we have as sequence of factors (there is no G action here)

$$(X, \nu) \to \cdots \to (X_n, \nu_n) \xrightarrow{\pi_n} (X_{n-1}, \nu_{n-1}) \to \cdots \to (X_1, \nu_1).$$

Now we can form the

$$L^\infty(X, \nu) \to \cdots \to L^\infty(X_n, \nu_n) \xrightarrow{\pi_{n*}} L^\infty(X_{n-1}, \nu_{n-1}) \to \cdots \to L^\infty(X_1, \nu_1).$$

where the maps π_{n*} are the conditional expectation. Given a function $f \in L^\infty(X, \nu)$ we can consider the sequence of functions $f_n \in L^\infty(X_n, \nu_n)$ which are the conditional expectations. So the sequence f_n forms a bounded martingale (since f, and hence all the rest, are bounded).

Since $\mathscr{B}_n \to \mathscr{B}$ it is natural to expect that $f_n \to f$ is some sense. It is not hard to see that $\|f - f_n\|_2 \to 0$, when we consider f_n as functions on X, by pulling them back. But in fact, f_n convergence to f is a stronger sense which is almost surely. The proof of that, which is classical result in probability is called the Martingale convergence theorem.

Theorem 4 — Martingale Convergence Theorem. Let X_n be a uniformly bounded martingale. Then X_n converges a.s. to a limit X with $\mathbb{E}(|X|) < \infty$.

In other words, for any $f \in L^1(X, \nu)$ and ν-almost every x, $f_n(x) \to f(x)$.

3. Stationary theory

3.1 Random Walks on Groups

For us, a *random walk* is just a measure $\mu \in \text{Prob}(G)$. We say that μ is *generating*, if the semigroup generated by the support of μ is the whole G. In terms of the random walk, it says that any element in G has a positive probability to be visited sometime by the random walk.

In other words, if μ is not generating, then the random walk will always miss some part of G. So we will always assume that μ is generating.

The distribution of the random walk ofter 2 steps is $\mu * \mu = \mu^2$, and the nth step is distributed according to μ^n. Formally, we cam consider the multiplication map $G \times G \to G$ given by $(g,h) \mapsto g \cdots h$. Then $\mu * \mu$ is the push forward of the measure $\mu \times \mu$ on $G \times G$ to G.

The generalization to μ^n is clear. Now μ is generating if and only if

$$\bigcup_{n \in \mathbb{N}} \text{Supp}(\mu^n) = G$$

which is equivalent to: $\text{Supp}(\mu)$ is a generating set of G.

Example 3.1 — Simple Random Walk on $\mathbb{Z}^d$.

Example 3.2 — Simple random walk on $\mathbb{F}_2$.

Example 3.3 — Lamplighter group. Let $G = \oplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}$. In other words, consider the group $\oplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. The group $\mathbb{Z}$ naturally acts on this group by automorphisms which is just acting on the index set: $zf(n) = f(n-z)$

The group $\oplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is call the *lamps group*.

Composition in this group is given by $(f_1, z_1)(f_2, z_2) = (f_1 + z_1 f_2, z_1 + z_2)$.

■ **Example 3.4 — $SL_2(\mathbb{Z})$.** Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then $< S,T >= SL_2(\mathbb{Z})$.

Consider the simple random walk. Can you imagine where a typical random walk is going to?

3.2 Stationary measures

Definition 10 A measured G-space (X,ν) is a (G,μ)**-stationary** or just (G,μ)-space, if it is a G-space and $\nu = \mu * \nu$. That is, for any $f \in L^\infty(X,\nu)$, $\int_X f(x)d\nu(x) = \sum_g \mu(g)\int_X f(gx)d\nu(x)$.

■ **Example 3.5** If $G \curvearrowright (X,m)$ is a measure preserving action then m is μ-stationary for any μ.

■ **Example 3.6 — $\mathbb{F}_2$ acting on $\partial\mathbb{F}_2$.** Let $\partial\mathbb{F}_2$ be the space of all infinte reduced words in the letters a,b,a^{-1},b^{-1}. This space is a compact space when equipped with the product topology. This topology is metrizable and a commonly used metric is given by $d(x,y) = \frac{1}{r}$ where r is the first letter such that $x_r \neq y_r$.

The group $\mathbb{F}_2$ acts on $\partial\mathbb{F}_2$ by adding the finite word at the beginning, and canceling if there is a need. It is not hard to check that this action is continuous and that $\partial\mathbb{F}_2$ has no invariant measure.

To define a measure on $\partial\mathbb{F}_2$ it is enough to say what are the measures of cylinders. Let w be is a finite reduced word. We denote by $[w]$ the set of all points in $\partial\mathbb{F}_2$ which start with w. Consider the following measure $\nu\,\mathrm{Prob}(\partial\mathbb{F}_2)$. $\nu([w]) = \frac{1}{4\cdot 3^{n-1}}$ where n is the length of the word w.

Then, for any $[w]$,

$$\mu * \nu([w]) = \frac{3}{4}\frac{1}{4\cdot 3^n} + \frac{1}{4}\frac{1}{4\cdot 3^{n-2}} = \frac{1}{4}\left(\frac{1}{4\cdot 3^{n-1}} + \frac{3}{4\cdot 3^{n-1}}\right) = \frac{1}{4\cdot 3^{n-1}} = \nu([w])$$

Lecture 5

3.2.1 The Markov Operator and properties of stationary measures

Fix a measure $\mu \in \mathrm{Prob}(G)$, and let $G \curvearrowright X$ be a continuous action. The Markov operator $M_\mu : \mathscr{C}(X) \to \mathscr{C}(X)$ is define by $M_\mu(f)(x) = \sum_g \mu(g)f(gx)$. The dual operator, $M_\mu^* : \mathrm{Prob}(X) \to \mathrm{Prob}(X)$ is $M_\mu^*(\theta) = \sum_g \mu(g)g\theta$.

Stationary measures are exactly the fixed points of the dual operator M_μ^*.

Remark 4 If ν is μ-stationary then it is also μ^n-stationary:

$$(\mu * \mu) * \nu = \mu * (\mu * \nu) = \mu * \nu = \nu$$

With less symbols - let (X,ν) be a stationary space, and find a compact model of X. Now ν is a fixed point of M_μ^*, and hence also it is a fixed point of any power of it.

Lemma 2 Any ergodic μ-stationary measure which is atomic-free, unless the space is finite and is invariant.

Proof. Assume that there is an atom x with maximal ν measure for a stationary measure ν. The orbit Gx is G-invariant with positive measure and by ergodicity $\nu(Gx) = 1$. So we deal with a countable space X equipped with a transitive G-action.

Since ν is stationary, $\nu(x) = \sum_g \mu(g)\nu(gx)$ but since $\nu(x)$ is the maximal value, $\nu(x) = \nu(gx)$ for every $g \in \mathrm{Supp}(\mu)$. Since ν is μ^n-stationary for any n, it holds for any $g \in \bigcup_n \mathrm{Supp}(\mu^n) = G$.

It follows that the measure ν is the uniform measure on the finite set Gx, hence, in particular it is invariant. ■

Lemma 3 Every μ-stationary action is a non-singular action (where as usual, μ is a generating measures).

Proof. We need to show that $\nu(A) = 0$ if and only if $g\nu(A) = 0$ for all $g \in G$. Assume that $\nu(A) = 0$. Then

$$0 = \nu(A) = \mu * \nu(A) = \sum_{g} \mu(g)\nu(A)$$

so $g\nu(A) = 0$ for every $g \in \mathrm{Supp}(\mu)$.

For other g's apply this argument on μ^n such that $g \in \mathrm{Supp}(\mu^n)$. Note that for any g such n exists by the generating assumption. ■

Corollary 3.2.1 The support of a stationary measure is an invariant set.

Lemma 4 Every compact G-space admits a stationary measure.

Proof. This follows by the amenability of $\mathbb{N}$, when thinking of the action of the dual Markov operator as an $\mathbb{N}$-action.

Take some arbitrary $\theta \in \mathrm{Prob}(X)$ and consider the sequence $\nu_n = \frac{1}{n}\left(\theta + \mu * \theta + \cdots + \mu^{n-1} * \theta\right)$. The compactness assets that there exists a limiting measure ν (for a subsequence).

$$\begin{aligned} \mu * \nu_n - \nu_n &= \frac{1}{n}(\mu * \theta + \cdots + \mu^n * \theta) - \frac{1}{n}\left(\theta + \mu * \theta + \cdots + \mu^{n-1} * \theta\right) \\ &= \frac{1}{n}(\mu^n * \theta - \theta) \to 0 \end{aligned}$$

which shows that any accumulation point of $\{\nu_n\}$ is μ-stationary.

Note: if X is metrizable, so the topology is separable. It means that the Borel structure is separable and then measure, in particular, the stationary that we found is standard. If X is crazy, ν is not necessarily standard. ■

A topological action $G \curvearrowright X$ is a **minimal** if X has no G-invariant subset, or equivalently, every orbit is dense.

Corollary 3.2.2 Let $G \curvearrowright X$ be a continuous action, and assume that X has a unique stationary measure. Then X has at most one minimal component.

Proof. The support of the stationary measure is the unique minimal component, since in each minimal component there is a stationary measure. ■

This shows how the existence of stationary measures can be used to study the topological action. Note that for amenable groups, the same is true for invariant measures, when the group is amenable.

Remark about amenability

In classical ergodic theory, the study of limiting behaviors of an action G goes though the understanding of invariant measures. However, invariant measures are not always exist. In facts, a group G that any compact G-space admits an invariant measure is called **amenable**. Hence, the classical theory of a single transformation ($\mathbb{Z}$ or $\mathbb{R}$ action) was generalized to the action of amenable groups during the 20th century.

Definition 11 A group G is **amenable** if there exist an invariant measure on any compact G-space.

By definition, a non amenable group admits a compact G-space the classical theory to these actions. In some sense, stationary measures without any invariant measure, and it is not clear how to generalize are good generalizations of invariant measures: there always exist, and yet they are meaningful.

Claim 1 Let G be a non-amenable group. Then for any generating μ, there are μ-stationary measures which are not invariant.

Remark 5 In fact, this characterizes amenability: a group is amenable if and only if there exists a generating measure μ such that all μ-stationary measures are already invariant.

The question what groups have the property that every μ-stationary measure is invariant for *every* μ is still open. We will see soon examples of such groups.

Lecture 6

3.2.2 Kakutani ergodic theorem

Given X be a measurable G-space, we can consider the space $X\times G^{\mathbb{N}}$, and the skew-product map $T : X\times G^{\mathbb{N}} \to X\times G^{\mathbb{N}}$, defined by $T(x,(g_1,g_2,\dots)) = (g_1x,(g_2,g_3,\dots))$. Then T is a non-invertible measurable map.

Given a probability measure ν on X we equipped with the probability measure $\mu^{\mathbb{N}}\times\nu$.

Lemma 5 The measure ν is μ-stationary if and only if $\mu^{\mathbb{N}}\times\nu$ is T-invariant.

Proof. Consider the factor map (which is not a G-map!) $\varphi : X\times G^{\mathbb{N}} \to X$. So $\varphi*(\nu\times\mu^{\mathbb{N}}) = \nu$. For any $\bar{f}\in L^\infty(X\times G^{\mathbb{N}},\nu\times\mu^{\mathbb{N}})$, let $f\in L^\infty(X,\nu)$ be $f=\varphi_*(\bar{f})$. We get that

$$\nu(f) = \nu(\varphi_*\bar{f}) = \varphi_*(\nu\times\mu^{\mathbb{N}})(\varphi_*\bar{f})) = \nu\times\mu^{\mathbb{N}}(\bar{f}).$$

Also,

$$\begin{aligned}\mu*\nu(f) &= \sum\mu(g)\int f(gx)d\nu(x) = \sum\mu(g)\int_{X\times G^{\mathbb{N}}}\bar{f}(gx,(g_2,\dots))d\nu\times\mu^{\mathbb{N}}(x,(g_2,\dots)) = \\ &= \sum\mu(g)\int_{X\times G^{\mathbb{N}}}\bar{f}(T(x,(g,g_2,\dots)))d\nu\times\mu^{\mathbb{N}}(x,(g_2,g_3,\dots)) \\ &= \int_{X\times G^{\mathbb{N}}}\bar{f}(T(x,(g,g_1,g_2,\dots)))d\nu\times\mu^{\mathbb{N}}(x,(g,g_2,g_3,\dots)) \\ &= T_*(\nu\times\mu^{\mathbb{N}})(\bar{f}).\end{aligned}$$

It follows that $\nu\times\mu^{\mathbb{N}}$ is T-invariant if and only if ν is μ-stationary

■

Theorem 3.2.3 Let (X,ν) be μ-stationary space. The following are equivalent:

1. The measures ν is ergodic
2. The Markov operator M_μ on $L^p(X,\nu)$, for all $1\le p\le\infty$ is ergodic. That is, the only P_μ-invariant functions are the constant functions.

3. The skew-product $(X \times G^{\mathbb{N}}, \nu \times \mu^{\mathbb{N}}, T)$ is ergodic.

Proof. (1) $\implies$ (2): For a given M_μ-invariant function $f \in L^p(X,\nu)$, we show that set on which $f(x) \geq 0$ is a G-invariant set, hence has trivial measure. By repeating this argument on $f-c$ for any c we get the function f is constant.

So let f be M_μ-invariant, and assume further, that $f^+(x) = \max\{f,0\}(x)$ is also M_μ-invariant. Let $E \subset X$ be a set with $\nu(E) = 1$ on which both f and f^+ are defined. Since the group g is countable, the set $E' = \bigcap_g gE$ is still of measure 1 (here we use that ν is non-singular). Now let $P' = \{x \in E' | f(x) \geq 0\}$. Every $x \in P'$ satisfies two equations: $f(x) = \sum \mu(g) f(gx)$ and also $f(x) = \sum \mu(g) f^+(gx)$. So $\sum \mu(g)(f^+(gx) - f(gx)) = 0$ and since these are non negative terms, $f^+(gx) = f(gx)$ for any $x \in P'$ and $g \in \mathrm{Supp}(\mu)$. Meaning that P' is g invariant for every $g \in \mathrm{Supp}(\mu)$. Since the support contains a generating set, P' is actually G-invariant, and by ergodicity has a trivial ν-measure.

Now we show that f^+ in M_μ-invariant. Since $\max\{f(x),0\} = \frac{f(x)+|f(x)|}{2}$, we only need to show that $|f|$ is also M_μ.

Note that $|f|(x) \leq |M_\mu(f)|(x)$ and so $M_\mu(|f|) - |f|$ is a non-negative function in $L^p(X,\nu)$. Now,

$$\int_X M_\mu(|f|) - |f| d\nu = \int_X M_\mu |(f)| d\nu - \int_X |f| d\nu$$

Since ν is μ-stationary, that is $(M_\mu)^*$-invariant, this expression equals 0. As it is a non negative function with 0 integral, we conclude that for ν-almost every x, $M_\mu(|f|)(x) = |f|(x)$.

(2) $\implies$ (3): Assume that M_μ is ergodic and let $f \in L^\infty(X \times G^{\mathbb{N}})$ be T-invariant. For any n consider $\mathscr{B}_n$ the sigma algebra on $G^{\mathbb{N}}$ generated by the first n letters. So given f we can push it to get $\overline{f_n} = \mathbb{E}(f|\mathscr{B}_n)$. As the σ-algebras are nested, we get that $\overline{f_n} = \mathbb{E}(f|\mathscr{B}_n) = \mathbb{E}(\mathbb{E}(f|\mathscr{B}_{n+1})|\mathscr{B}_n) = \mathbb{E}(\overline{f_{n+1}}|\mathscr{B}_n)$.

As the functions $\overline{f_n}$ depend only on the X coordinate and the first n coordinate of $G^{\mathbb{N}}$, we can define functions f_n on $X \times G^n$ by

$$f_n(x,(g_1,g_2,\dots g_n)) = \overline{f_n}(x,(g_1,g_2,\dots,g_n)) = \int f(x,g_1,g_2,\dots) d\mu^{\mathbb{N}}(g_{n+1},g_{n+2},\dots).$$

Note that if we feed f_n with a sequence $g_1,g_2,\dots g_n$ which cannot happen in the random walk, say if $g_1 \notin \mathrm{Supp}(\mu)$ then f_n would give 0.

Since f is T-invariant, we get that for any n,

$$\begin{aligned} f_n(x,(g_1,g_2,\dots g_n)) &= \int f(x,(g_1,g_2,\dots)) d\mu^{\mathbb{N}}(g_{n+1},g_{n+2},\dots) \\ &= \int f(g_1x,(g_2,\dots)) d\mu^{\mathbb{N}}(g_{n+1},g_{n+2},\dots) \\ &= f_{n-1}(g_1x,(g_2,g_3,\dots g_n)) \end{aligned}$$

and so

$$f_n(x,(g_1,g_2,\dots g_n)) = f_0(g_n g_{n-1} \cdots g_1 x)$$

In particular, for $n=1$ we get that

$$f_0(x) = \mathbb{E}(\overline{f_1}|\mathscr{B}_0)(x) = \sum \mu(g_1) f_1(x,(g_1)) = \sum \mu(g_1) f_0(g_1x) = M_\mu(f_0)(x).$$

Since M_μ is ergodic, $f_0 = c$ is constant. Since $\overline{f_1} = \mathbb{E}(\overline{f_2}|\mathscr{B}_1)$ the same argument shows that f_1 is constant function. As $f_0(x) = \mathbb{E}(\overline{f_1}|\mathscr{B}_0)(x)$ we conclude that in fact $f_1 = f_0 = c$. The applies to any n, so we get that $f_n = c$ for any n.

Now by the Martingale convergence theorem, f is also the constant function c. In fact, we don't need the almost surely convergence result, and the soft, L^2 result is enough. Since we know that all the projections are constant, so $\|f - f_n\|_2 \to 0$ says that $\|f - c\| = 0$ hence $f = c$ almost surely.

(3) $\implies$ (2): Any M_μ-invariant function in $L^p(X,\nu)$ can be lifted to an L^p-function on the skew product, which is T-invariant and constant on $G^{\mathbb{N}}$, which shows that the skew product is not ergodic.

(2) $\implies$ (1): Any G-invariant set is M_μ-invariant, hence the existence of a G-invariant of non-trivial measure, implies that M_μ is not ergodic. ■

Lecture 7

Corollary 3.2.4 — Kakutani ergodic theorem. Let (X,ν) be an ergodic μ-stationary action. Then for any $f \in L^1(X,\nu)$ and $\mu^{\mathbb{N}}$-almost every $(g_1, g_2, g_3, \dots)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(g_k g_{k-1} \cdots g_2 g_1 x) = \int_X f(x) d\nu(x)$$

Proof. By the previous theorem we get that $G^{\mathbb{N}} \times \nu$ is T-ergodic measure. Take a function $f \in L^1(X,\nu)$ and lift it to a function $\bar{f}(x,(g_1,g_2,\dots)) = f(x)$, and apply the classical Birkhoff ergodic theorem to get

$$\begin{aligned}
\int_X f(x) d\nu(x) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \bar{f}(T^k(x,(g_1,g_2,\dots))) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \bar{f}(g_k g_{k-1} \cdots g_2 g_1 x, (g_{k+1}, g_{k+2}, \dots)) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(g_k g_{k-1} \cdots g_2 g_1 x)
\end{aligned}$$

■

Remark 6 As invariant measures are stationary, this works also for invariant measures!

It is in particular interesting when the group is non-amenable, where there is a difficulty to have a natural way to take averages.

Corollary 3.2.5 Let X be compact, and let $\nu \in \mathrm{Prob}(X)$ be an ergodic stationary measure. If η is stationary ergodic and $\eta << \nu$ then $\eta = \nu$.

Proof. Let $f \in C(X)$ be some function. Assume that $\eta << \nu$, and let $E \subset X$ be such that $\eta(E) > 0$ and hence also $\nu(E) > 0$. So there are $E_1 \subset E$ with $\nu(E_1) = \nu(E)$ and $\Omega_1 \subset G^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(\Omega_1) = 1$ such that for any $x \in E_1$ and $(g_1, g_2, \dots) \in \Omega_1$, the ergodic theorem holds for ν.

Find $E_2 \subset E$ with $\eta(E_2) = \eta(E)$ and Ω_2 with $\mu^{\mathbb{N}}(\Omega_2) = 1$ for which the theorem holds for η. Now apply for any point in $E' = E_1 \cap E_2$ and $\Omega' = \Omega_1 \cap \Omega_2$ to get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(g_k g_{k-1} \cdots g_2 g_1 x) = \int_X f(x) d\nu(x) = \int_X f(x) d\eta(x).$$

■

Let X be a compact G-space. Denote by $\mathrm{Prob}_\mu(X)$ to be the set of all μ-stationary measures on X. Then $\mathrm{Prob}_G(X) \subset \mathrm{Prob}_\mu(X) \subset \mathrm{Prob}(X)$.

Exercise 3.1 $\text{Prob}_\mu(X)$ is a compact convex space.

Solution Show that the convolution operator (the dual of the Markov operator) is a continuous function. ■

Corollary 3.2.6 A μ-stationary is ergodic if and only if it is an extremal point in $\text{Prob}_\mu(X)$.

Proof. If A is an invariant set of non-trivial ν-measure, then the we can express ν as a convex combination of the (normalized) restriction to A and the (normalized) restriction to the complement of A (one should verify that these are indeed stationary measures).

Now assume that ν is ergodic, and assume that $\nu = c\eta + (1-c)\theta$ where η and θ are ergodic, stationary and $0 < c < 1$. So $\eta << \nu$ in contradiction. ■

Here is the topological analogue of that: a topological action $G \curvearrowright X$ is said to be **minimal** if every x, has a dense orbit (that is, $\overline{\{G \cdot x\}} = X$). The content of the following theorem that walks from $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ can be taken to revel the density.

Theorem 3.2.7 Let $G \curvearrowright X$ be a continuous action on a compact metric space, and let $\mu \in \text{Prob}(G)$ be generating. Then for $\mu^{\mathbb{N}}$-almost every walk $g_1, g_2, \ldots$, and every x, the set $\{g_n g_{n-1} \cdots g_2 g_1 x\}$ is dense in X.

TODO: write a proof

Proof. ■

3.2.3 Random Walks from dynamical perspective

In probability theory it is common to this of a random walk as a sequence of random variables. Sometimes it is useful to consider it in this way, and sometimes we will prefer the The space of walks and the Markovian measure.

Consider $G^{\mathbb{N}}$, the space of all sequences in G equipped with the product sigma algebra. Consider the multiplication map $m : G^{\mathbb{N}} \to G^{\mathbb{N}}$ defined by $(m(w))_n = w_1 \cdot w_2 \cdots w_n$. The push-forward measure $\mathbb{P}_\mu = m_* \mu^{\mathbb{N}}$ is called the Markovian measure.

When $G^{\mathbb{N}}$ is equipped with the product measure $\mu^{\mathbb{N}}$ then the coordinates represents the increments of the random walk.

When equipped with $\mathbb{P}_\mu$, we think of $G^{\mathbb{N}}$ is the space of walks of the random walk, as the w_n is the position of the random walk at time n. To distinguish between the two, we denote the space of walks by $(\Omega, \mathbb{P}_\mu)$.

For example, consider the set of all paths ω such that e appears in infinitely many coordinates. If the measure of this set is 1 then the random walk is recurrent.

TODO: add: $(\Omega, \mathbb{P}_\mu)$ is a G-space, while $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ is not

3.3 Harmonic Functions

Definition 12 A function $h : G \to \mathbb{R}$ is μ-harmonic if

$$h(g) = \sum_\gamma \mu(\gamma) h(g\gamma).$$

Exercise 3.2 Find all μ-harmonic functions (bounded and unbounded) on $\mathbb{Z}$ where μ is the simple random walk on $\mathbb{Z}$. Conclude that the only bounded harmonic functions are the constant functions.

Exercise 3.3 Find a harmonic function for the simple random walk on F_2.

We will be only interested here in bounded harmonic functions, that is functions in $\ell^\infty(G)$ which are harmonic. Denote the Banach space of all bounded harmonic functions by $H^\infty(G,\mu)$. Note that $H^\infty(G,\mu)$ is never empty - it at least contains the constant functions. So $H^\infty(G,\mu)$ is s sub vector space of the Banach space $\ell^\infty(G)$. It is also topologically closed. In other words, the sup norm on $H^\infty(G,\mu)$ induces a complete topology, so it is a Banach space.

Note that $\ell^\infty(G)$ has another structure - it is a algebra, when considering the pointwise multiplication. Pointwise multiplication of harmonic functions is not harmonic, so $H^\infty(G,\mu)$ is a sub-Banach space, but not a sub-algebra.

Let $h \in H^\infty(G,\mu)$ be a bounded harmonic function.

Lemma 6 The limit $\lim_{n\to\infty} h(w_n)$ exists for $\mathbb{P}_\mu$-a.e. $w \in \Omega$. So any $h \in H^\infty(G,\mu)$ corresponds to some $\tilde{h} \in L^\infty(\Omega,\mathbb{P}_\mu)$, and

$$h(e) = \int_\Omega \tilde{h}(w)d\mathbb{P}_\mu(w).$$

Moreover, for a fixed g the limit $\lim_{n\to\infty} h(gw_n) = \tilde{h}(g,w)$ exists and

$$h(g) = \int_\Omega \tilde{h}(g,w)d\mathbb{P}_\mu(w).$$

Proof. Let $\mathscr{B}_n$ be the sigma algebra generated by the first n letters in Ω, including $\mathscr{B}_0$ which is the trivial σ-algebra. Then $\mathscr{B}_n \to \mathscr{B}$. Now define $f_n(w) = h(w_n)$ and $f_0 = h(e)$. Then $\{f_n\}$ forms a bounded martingale:

$$\mathbb{E}(f_{n+1}|\mathscr{B}_n)(w) = \sum_G \mu(\gamma)h(w_n\gamma) = h(w_n) = f_n(w).$$

By the martingale it converges for a.e. $w \in \Omega$. Once we know that $\tilde{h}$ exists, then $h(e)$ is the conditional expectation of $\tilde{h}$, with respect hence the equality.

For the “moreover“, fix g and define $f_n(w) = h(gw_n)$ (and $f_0 = h(g)$). ■

Remark 7 The meaning of the preceding lemma, is that if one want to understand something about limiting behavior of a random walk - it is useful to consider bounded harmonic functions as they have an opinion of almost every walk.

Some idea is to say that two walks convergence to the same “direction” if all the bounded harmonic functions cannot distinguish between them. That is, define an equivalence relation on walks, $w \sim w'$ if $\lim h(w_n) = \lim h(w'_n)$ for all h. However, maybe there are other functions that have an opinion? We will deal with this questions when we will construct the Furstenberg-Poisson boundary.

Where do harmonic functions are coming from?

Lecture 8

3.3.1 The Poisson Transform

One of the most natural source of G-spaces, is spaces of functions over G. For $f : G \to X$ (where X is some set) we write $g.f(\gamma) = f^g(\gamma) = f(g\gamma)$. This is a right-action: $f^{g_1g_2}(\gamma) = f(g_1g_2\gamma) = f^{g_1}(g_2\gamma) = (f^{g_1})^{g_2}(\gamma)$ so as functions $f^{g_1g_2} = (f^{g_1})^{g_2}$.

More generally, if G acts on some space X, then G acts naturally on spaces of function over X. For example, consider $L^\infty(X,\nu)$, where X is a G-space, by again $g.f(x) = f^g(x) = f(gx)$.

Definition 13 Let (X,ν) be a G-space. The **Poisson transform** is the map $\mathscr{P}_\nu : L^\infty(X,\nu) \to \ell^\infty(G)$ given by $\mathscr{P}_\nu(f)(g) = g\nu(f) = \int_X f(gx)d\nu(x)$.

Lemma 7 The Poisson transform $\mathscr{P}_\nu : L^\infty(X,\nu) \to \ell^\infty(G)$ is a unital (sends the constant function 1 to 1), positive (sends non-negative function on X to non negative functions on G), linear and G-equivariant.

Proof. Linearity and positivity are clear from since it is defined by integral, and it is unital since ν and all the $g\nu$ are probability measures.

G-equivariance:

$$\mathscr{P}_\nu(f^\gamma)(g) = \int_X f^\gamma(gx)d\nu(x) = \int_X f(\gamma gx)d\nu(x) = \mathscr{P}_\nu(f)(\gamma g).$$

■

We just saw that any $\nu \in \mathrm{Prob}(X)$, where X is a compact G-sapce gives a unital positive linear map $C(X) \to \ell^\infty(G)$.

Lemma 8 Let $G \curvearrowright X$ be a continuous action, and let $\mathscr{P} : C(X) \to \ell^\infty(G)$ be a unital, positive and linear equivariant map. Then there exists some $\nu \in \mathrm{Prob}(X)$ such that $\mathscr{P}_\nu = \mathscr{P}$.

Hence we identified between $\mathrm{Prob}(X)$ and the space of unital positive linear equivariant maps.

Proof. For any continuous function $f \in C(X)$ define $\nu(f) = \mathscr{P}(f)(e)$. Note that ν is linear positive map on $C(X)$ with $\nu(1) = 1$ and hence is a probability measure. ■

It is clear that $H^\infty(G,\mu)$ is a sub vector space of $\ell^\infty(G)$, and we now observe that it is G-invariant. The key here is that the hamonicity condition is on the right side of the argument, while the action affects the left side:

Let $h \in H^\infty(G,\mu)$ and fix some $g_0 \in G$. Then

$$\sum_\gamma \mu(\gamma)h^{g_0}(g\gamma) = \sum_\gamma \mu(\gamma)h(g_0g\gamma) = h(g_0g) = h^{g_0}(g)$$

so h^{g_0} is also μ-harmonic, since it is clearly bounded, we get that so $G \curvearrowright H^\infty(G,\mu)$.

Lemma 9 Let (X,ν) be a G-space. The measure ν is μ-stationary if and only if the image of $\mathscr{P}_\nu$ is in $H^\infty(G,\mu)$.

The measure ν is invariant if and only if the image of $\mathscr{P}_\nu$ are the constant functions.

Proof. Let $f \in L^\infty(X,\nu)$, and let $h = \mathscr{P}_\nu(f)$. Then $h(g) = \mathscr{P}_\nu(f)(g) = g\nu(f)$ and

$$\begin{aligned}
\sum_\gamma \mu(\gamma)\cdot h(g\gamma) &= \sum_\gamma \mu(\gamma)\cdot \mathscr{P}_\nu(f)(g\gamma) \\
&= \sum_\gamma \mu(\gamma)\cdot g\gamma\nu(f) \\
&= g\left(\sum_\gamma \mu(\gamma)\cdot\gamma\nu\right)(f) \\
&= g\mu * \nu(f)
\end{aligned}$$

So h is harmonic if and only if $\mu * \nu = \nu$.
The statement about invariant measures is clear. ■

3.4 Choquet-Deny theorem

Theorem 5 — Choquet-Deny. Let G be an abelian group, and let $\mu \in \operatorname{Prob}(G)$ be some generating measure. Then $H^\infty(G,\mu)$ consists of only constant functions.

In particular, any μ-stationary measure, for any μ on an abelian group is invariant.

Proof. Let K be the set of all μ-harmonic function which are bounded between 0 and 1. K is closed in the space of all real functions on G so it is compact and convex. By Krein-Milman K is the closure of the convex hull of its extremal points, so it enough to show that that all the extremal functions are constant.

Recall that G acts on $H^\infty(G,\mu)$ from the left. Since G is abelian, the right G-action $h^g(\gamma) = h(g\gamma)$ also preserves harmonicity, hence preserves K. Let h be extremal in K. Then

$$h(g) = \sum_\gamma \mu(\gamma)\cdot h(g\gamma) = \sum_\gamma \mu(\gamma)\cdot h(\gamma g) = \sum_\gamma \mu(\gamma)\cdot h^\gamma(g)$$

and so as functions, $h = \sum_\gamma \mu(\gamma) h^\gamma$ but h is extremal - so $h^\gamma = h$ for all $\gamma \in \operatorname{Supp}(\mu)$. Since ν is μ^n stationary and μ is generating, h is a constant function. ■

Exercise 3.4 Let μ be a generating measure on a nilpotent group G. Show that any μ-stationary measure is invariant.

Hint: Show first that any bounded harmonic function must be constant on the center of G.

Solution Todo ■

3.5 Conditional Measures

Note that in the following we consider a **compact, metrizable** stationary space.

Lemma 10 Let X be a compact metrizable G-space, and let $\nu \in \operatorname{Prob}(X)$ be a μ-stationary measure. Then the limit $\lim_{n\to\infty} w_n\nu = \nu_w$ exists for a.e. $w \in \Omega$, $\nu = \int_\Omega \nu_w d\mathbb{P}_\mu(w)$.

Moreover, for any fixed g and $\mathbb{P}_\mu$-a.e. w the limit $\lim_{n\to\infty} gw_n\nu = \nu_{gw}$ exists, and $g\nu_w = \nu_{gw}$.

Proof. Since X is metrizable, $C(X)$ is separable, and let $f_1, f_2, \dots$ be a countable set of functions with $\|f_k\| = 1$ such that their span is dense in $C(X)$. For each f_k, there exists a subset $\Omega_k \subset \Omega$ with $\mathbb{P}_\mu(\Omega_k) = 1$ on which the harmonic function $\mathscr{P}_\nu(f)$ has a limit $\mathscr{P}_\nu(f)(w_n) = \lim w_n\nu(f)$. Let $\Omega_0 = \bigcap_k \Omega_k$ so it is a set of $\mathbb{P}_\mu$-measure one on which we can define for any $w \in \Omega_0$ a linear functional on the set $\{f_k\}$. These functionals are bounded, hence continuous and so they extend to linear functionals ν_w on $C(X)$. ■

Lecture 9

3.6 On the category of stationary actions

Lets discuss some functorial properties of stationary spaces. We saw that a factor of a stationary measure is stationary.

Lemma 11 A factor of a stationary space is a stationary space.

Proof. Let $(X,\nu) \xrightarrow{\pi} (Y,\eta)$ and assume that ν is μ-stationary. Then

$$\mu * \eta = \mu * \pi\eta = \pi(\mu * \nu) = \pi\nu = \eta.$$

■

Theorem 6 — (Naturality of the conditional measures. Let $\pi : (X,\nu) \to (Y,\eta)$ be a G-factor measurable map between two compact metrizable space. Then for $\mathbb{P}_\mu$-almost every w, $\pi_*\nu_w = \eta_w$.

In particular, the conditional measures are measurable object, and do not depend on the compact model, as one might think. Indeed, if (X,ν) and (Y,η) are two compact model of the same abstract stationary action, then the measurable isomorphism $\pi : X \to Y$ form an isomorphism of $\pi : (X,\nu_w) \to (Y,\eta_w)$ for $\mathbb{P}_\mu$-almost every w.

Note that this isomorphims is not a G-isomorphism (G doesn't act on a single conditional measure, but rather, takes one conditional to another one). Hence this isomorphism is as measure spaces, which is not a very strong condition. It does imply, for example, that the number of atoms, is the same, and this is independent on the compact model.

Remark 8 — Stationary measures as boundary maps. The existence of conditional measures, shows that a stationary measure on a compact metrizable space X can be thought of as a measurable maps $b : (\Omega, \mathbb{P}_\mu) \to \mathrm{Prob}(X)$. It is clear that b is shift invariant as ν_w is defined by a limit.

Since we know that any compact space admits a stationary measure, we get that such a map b is defined to $\mathrm{Prob}(X)$ for any compact metrizable G-space X.

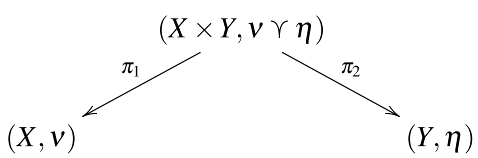
The naturality results shows that if we have a factor map $X \to Y$ and instead of thinking of stationary measures ν and η, but rather as maps b_X and b_Y, then $b_Y(w) = \pi(b_X(w))$.

We will discuss it further but already now we can observe some connection with amenability. Note that G acts on $(\Omega, \mathbb{P}_\mu)$ by "starting from g instead of e", and that b is equivariant. Also, on $(\Omega, \mathbb{P}_\mu)$ the shift acts, and b is actually shift invariant. Hence if $(\Omega, \mathbb{P}_\mu)$ is trivial after modding out by the shift, then we get an invariant measure on any compact space, which implies that the group is amenable!

If not, one can think of it as a nice replacement.

What about products? In general, if we have $G \curvearrowright X$ and $G \curvearrowright Y$ then the diagonal action $G \curvearrowright X \times Y$ is defined by $g(x,y) = (gx, gy)$.

Definition 14 Let (X,ν) and (Y,η) be two (G,μ)-stationary space. A **joining** of ν and η is a μ-stationary measure $\nu \curlyvee \eta \in \mathrm{Prob}(X \times Y)$ which ν and η as the projection. In a diagram,



This is closely related to the notion of *coupling* in probability theory.

Usually in ergodic theory when discussing invariant measures, a joining of invariant measures, is an invariant measure on the product which extends both measure. In the invariant measure setup, there is a natural joining which is the product measure $\nu \times \eta$. Indeed $g(\nu \times \eta) = g\nu \times g\eta = \nu \times \eta$.

In general, if we have two stationary actions, the diagonal action on the product space is not stationary anymore. The reason is, after thinking of the Poisson transform, that multiplication of harmonic functions is not harmonic. In any case, any two stationary actions can be joined. Choose a compact model for X and for Y, and consider the subspace of all measures in $\mathrm{Prob}(X \times Y)$ which projects on ν and η. This is a (weak*-)compact convex space hence the dual of the Markov operator has a fixed point there. More explicitly, start from some measure there, θ and take a weak*-limit of the averages of the convolutions of μ with θ.

But this is still quite abstract, the following is a hands-on construction when the stationary measures are standard. Recall that the compact model theorem says that when dealing with abstract standard space, one can always find a metrizable compact model, and hence to get conditional measures to work with.

The canonical joining

Any two stationary actions on standard spaces can be naturally joined using the following measure:

$$\lambda = \int_\Omega \nu_w \times \eta_w d\mathbb{P}_\mu(w).$$

Indeed, for each $g \in G$ we have

$$\begin{aligned} g\lambda &= \int_\Omega g\nu_w \times g\eta_w d\mathbb{P}_\mu(w) \\ &= \int_\Omega \nu_{gw} \times \eta_{gw} d\mathbb{P}_\mu(w) \\ &= \int_\Omega \nu_w \times \eta_w dg\mathbb{P}_\mu(w) \end{aligned}$$

where $(gw)_n = gw_n$. Hence we get that

$$\begin{aligned} \sum_g \mu(g) \cdot g\lambda &= \sum_g \mu(g) \cdot (\int_\Omega \nu_w \times \eta_w dg\mathbb{P}_\mu(w)) \\ &= \int_\Omega \nu_w \times \eta_w d\mathbb{P}_\mu(w) = \lambda. \end{aligned}$$

This construction becomes even more natural when considering stationary measure as boundary maps. Instead of thinking about the stationary measures on X and on Y, consider the maps $b_X : (\Omega, \mathbb{P}_\mu) \to X$ and $b_Y : (\Omega, \mathbb{P}_\mu) \to Y$. The first boundary map to $X \times Y$ that comes to mind is $b_X \times b_Y : (\Omega, \mathbb{P}_\mu) \to X \times Y$, which gives the measure λ above.

Under some conditions on the stationary spaces, this natural joining is also the unique one, but in general, there are many joinings of given two stationary measures.

Lemma 12 If ν is measure preserving then $\nu \curlyvee \eta = \nu \times \eta$.

Proof.

$$\begin{aligned} \nu \curlyvee \eta &= \int_\Omega \nu_w \times \eta_w d\mathbb{P}_\mu(\omega) \\ &= \int_\Omega \nu \times \eta_w d\mathbb{P}_\mu(w) \\ &= \nu \times \int \eta_w d\mathbb{P}_\mu(w) \\ &= \nu \times \eta. \end{aligned}$$

∎

4. Boundary theory

4.1 Compact μ-boundaries

Definition 15 A compact (G,μ)-stationary, (B,ν) is called **a compact-(G,μ)-boundary** if $\nu_w = \delta_{\mathbf{bnd}}(w)$ is a Dirac measure for a.e. $w \in \Omega$.

Recall that a given a stationary measure ν on metrizable compact spaces yields to a boundary map $\mathbf{bnd} : (\Omega, \mathbb{P}_\mu) \to \mathrm{Prob}(X)$. If ν happen to be μ-boundary, then we get a map $\mathbf{bnd} : (\Omega, \mathbb{P}_\mu) \to X$, and $\mathbf{bnd}_* \mathbb{P}_\mu = \nu$.

Example The trivial space (one point space), is a μ-boundary for any μ on any G.

Example The free group.

We need to show that for any set, and almost every w, $w_n\nu(E) \to \{0,1\}$ It is enough to show it for cylinder sets. So let $[u]$ be a cylinder, so u is a word of length m. Since the simple random walk is transient, for almost every w, there exists a time $t(w)$ when the random walk won't visit the ball of radius $2m$ anymore. We get a map, defined almost everywhere, $t : (\Omega, \mathbb{P}_\mu) \to \mathbb{N}$ such that the $2m$-prefix in w_n, for all $n > t(w)$ are the same word, say v. Now

$$w_n\nu([u]) = \nu((w_n)^{-1}[u]) = \nu(g_n^{-1} \cdots g_{m+1}^{-1} v^{-1}[u])$$

Recall that $\nu \in \mathrm{Prob}$

Lemma 13 Compact boundaries are ergodic.

Proof. Let $E \subset B$ be an invariant set. Then

$$\nu_w(E) = \lim_{n\to\infty} w_n \nu(E) = \lim_{n\to\infty} \nu\left((w_n)^{-1} E\right) = \nu(E)$$

And since ν_ω is a Dirac measure, $\nu(E)$ is either 0 or 1. ■

Lemma 14 Let (B,ν) be a compact boundary. Then the Poisson transform $\mathscr{P}_\nu : L^\infty(B,\nu) \to \ell^\infty(G)$ is an isometry onto its image.

Proof. Note that any non-trivial μ-boundary satisfies the following property (known as the SAT property): For any E with $\nu(E) > 0$ and $\varepsilon > 0$, there exists some g such that $\nu(gE) > 1-\varepsilon$.

Indeed, since $\nu = \mathbf{bnd}_* \mathbb{P}_\mu$ we get that the preimage of E is a positive measure set of walks for which every w, $\lim w_n \nu(E) = \delta_{\mathbf{bnd}(w)}(E) = 1$.

Now for characterstic functions, it follows that $\left\|\mathscr{P}_\nu(1_E)\right\|_\infty = 1$. As the chracteristic functions span a dense set in $L^\infty(B,\nu)$, we conclude that $\mathscr{P}_\nu$ is an isometry. ■

Lemma 15 A measurable G-factor of a compact boundary is a compact boundary.

Proof. Let $(X,\nu) \xrightarrow{\pi} (Y,\eta)$ both are compact, where (X,ν) is a boundary. Then $\eta_w = \pi(\nu_w)$ for $\mathbb{P}_\mu$-almost every w. Since the image of Dirac is Dirac, η_w are Dirac measures almost surely. ■

The name boundary comes from the idea that we can attached (B,ν) to the group G: Consider the space $G \cup B$, with a topology, that a sequence ω in G convergences to $b \in B$ if $\operatorname{bnd}(\omega) = b$.

Claim 2 If (X,ν) is a (G,μ)-boundary and ν is an invariant measure then (X,ν) is trivial.

Proof. Since ν is invariant, $\nu_\omega = \nu$, and since (X,ν) is a boundary, $\nu = \nu_\omega$ is a Dirac measure. ■

Any μ-boundary gives some information on the μ-random walk - if $\operatorname{bnd}(\omega_1) \neq \operatorname{bnd}(\omega_2)$ then we think of the walks as converging to different points at infinity. The trivial boundary does not distinguish between the walks - they all goes to infinity.

Thus, we will be interested in finding the largest boundary possible. This boundary is the Furstenberg-Poisson boundary.

4.1.1 Joining of compact boundaries

Lemma 16 If (X,ν) is a compact (G,μ)-boundary then $\nu \curlyvee \eta$ is the unique joining of ν and η.

Proof. Let λ be a some joining of ν and η. Then the conditional measures λ_w have marginals $\pi_{1*}(\lambda_w) = \delta_{b(w)}$ and $\pi_{2*}(\lambda_w) = \eta_w$. Since $\pi_1(\lambda_w)$ is a Dirac measure, we get that $\lambda_w = \delta_{b(w)} \times \eta_w$ and so

$$\lambda = \int_\Omega \lambda_w d\mathbb{P}_\mu(w) = \int_\Omega \delta_{b(w)} \times \eta_w d\mathbb{P}_\mu(w) = \nu \curlyvee \eta.$$

■

We get now a strong ergodic property of boundaries:

Corollary 1 Let (X,ν) be a μ-boundary, and let (Y,m) be an ergodic measure preserving action. Then $(X \times Y, \nu \times m)$ is ergodic.

Proof. Pick compact models and consider the subspace K in $\operatorname{Prob}(X \times Y)$ consists of all stationary measures which projects on ν and m respectively. Then K is compact and convex.

We claim that a measures in an extremal point in K if and only if it is ergodic.

Now since ν is a compact μ-boundary, there is a unique stationary joining, hence K is a single point, which must be extremal and hence ergodic. ■

If the measure ν was an invariant measure, rather than stationary, then such a measure is called weakly mixing.

Corollary 2 The only joining of a measure preserving system and a boundary is the product. It means that in a sense, these properties are orthogonal to each other.

Corollary 3 The joining of two compact (G,μ)-boundaries is a compact (G,μ)-boundary.

Corollary 4 There exists a universal (G,μ) boundary (Π,ν): every (G,μ)-boundary is a factor if (Π,ν). The universal boundary called the Furstenberg-Poisson boundary.

4.2 The Furstenberg-Poisson boundary

Definition 16 An abstract (G,μ)-space, (B,ν) is called a (G,μ)-boundary if there exist a compact model of (B,ν) which is a compact boundary.

Theorem **[Furstenberg '73]** Let (G,μ) be a locally compact second countable group with admissible measure. There exists a uniquely defined universal boundary called the Poisson boundary, (Π,ν), in the sense that any other boundary is a G-factor of (Π,ν).

By uniqueness we mean: any two universal boundaries are G-isomorphic.

4.2.1 Construction via harmonic functions

Real commutative C^*-algebras

Let $\mathscr{A}$ be a unital commutative real C^*-algebra is a commutative Banach algebra. That is, $\mathscr{A}$ is a normed linear space over $\mathbb{R}$ with a commutative multiplication and a unit element and $\left\|x^2\right\| = \|x\|^2$ for $x \in \mathscr{A}$.

An example of such an object is $\mathscr{C}(K)$ where K is compact Hausdorff space (note that not necessarily metrizable).

This is indeed the only the source for examples: **Gelfand-Naimark theorem** relates these two categories, of real C^*-algebra and compact spaces. More precisely, it says that for every such algebra $\mathscr{A}$ there exists a compact space K such that $\mathscr{C}(K)$ isometrically $*$-isomorphic to $\mathscr{A}$ (when $\mathscr{C}(K)$ equipped with the sup norm: $\|f\|_\infty = \sup_{k\in K}|f(k)|$). The algebra $\mathscr{A}$ is separable if and only if K is metrizable. K is called **the spectrum or the Gelfand space of $\mathscr{A}$**.

This correspondence between the categories of C^*-algebras and compact spaces, goes through as follows:

There is a correspondence between closed sub algebra $\mathscr{S} \leq \mathscr{A}$ (containing the unit) and continuous onto maps $K_1 \to K_2$ where K_1 and K_2 are the spectrums of $\mathscr{A}$ and $\mathscr{S}$ respectively.

Stone-Banach theorem

Another theorem that we will use is Stone-Banach. Let X,Y compact spaces. If $\pi : X \to Y$ is a homemorphism then it defines a function $\pi^* : \mathscr{C}(Y) \to \mathscr{C}(X)$ by $\pi^*(f)(x) = f(\pi(x))$. Note that π^* is an isomerty and $\pi^*(1_Y) = 1_X$.

Stone-Banach gives the other direction: if $\varphi : \mathscr{C}(Y) \to \mathscr{C}(X)$ is an surjective linear isomerty $\varphi(1_Y) = 1_X$ then it is induced by a homeomorphism $\pi : X \to Y$.

We will use it as follows: if G acts on $\mathscr{C}(X)$ by isometries then G acts on X.

4.2.2 The construction of Π_{top}

To avoid some technicalities, we will prove it to discrete G.

Recall that the Martingale convergence theorem asserts that any bounded harmonic function has an opinion on almost every walk $\omega \in (G,\mu)^{\mathbb{N}} = (\Omega,\mathbb{P})$ and we asked whether there are more functions with this property.

Definition Denote by $B(G)$ the Banach the space of all bounded functions on G.

Let $\mathscr{A}$ be the algebra of all $f \in B(G)$ such that the limit $\lim_{n\to\infty} f(gw_n) = \tilde{f}(g,w)$ exists for every $g \in G$ and $\mathbb{P}$-a.e. $\omega \in \Omega$.

- We saw that $\mathscr{H} = \mathscr{H}^\infty(G,\mu) \subseteq \mathscr{A}$.

- Unlike $\mathscr{H}$, $\mathscr{A}$ is an algebra: the pointwise multiplication preserves the existence of the limit a.e., the norm is the sup norm.
- $\mathscr{A}$ This is also a G-space: the action $f^g(\gamma) = f(g\gamma)$ preserves the condition that the limit exists for all g and a.e. ω.

Let $\mathscr{I}$ be the set of all functions in $\mathscr{A}$ that the limit exists and equal to zero for every $g \in G$ and $\mathbb{P}$-a.e. $\omega \in \Omega$.

- $\mathscr{I}$ is an ideal.
- The G-action preserves $\mathscr{I}$.

Thus, G acts on the C^*-algebra $\mathscr{A}/\mathscr{I}$!

Lemma $\mathscr{A} = \mathscr{H} \oplus \mathscr{I}$.

Proof. First, $\mathscr{H} \cap \mathscr{I} = \{0\}$ since we saw that $h(g) = \int_\Omega \tilde{f}(g,\omega)d\mathbb{P}_\mu(w)$ so if $\tilde{f}(g,w) = 0$ for almost every w then $h(g) = 0$.

Given $f \in \mathscr{A}$ define $h_f(g) = \int \tilde{f}(g,w)d\mathbb{P}_\mu(w)$. So h_f is harmonic:

$$\begin{aligned}
\sum_{\gamma\in G} \mu(\gamma) h_f(g\gamma) &= \sum_{\gamma\in G} \mu(\gamma) \int_\Omega \tilde{f}(g\gamma,w)d\mathbb{P}_\mu(w) \\
&= \sum_{\gamma\in G} \mu(\gamma) \int\limits_\Omega \tilde{f}(g,\gamma w)d\mathbb{P}_\mu(w) \\
&= \int\limits_\Omega \tilde{f}(g,w)d\mathbb{P}_\mu(w) = h_f(g)
\end{aligned}$$

$f - h_f \in \mathscr{I}$:

$$\begin{aligned}
h_f(gw_n) &= \int\limits_\Omega \tilde{f}(gw_n,w')d\mathbb{P}_\mu(w') \\
&= \mathbb{E}\left(\tilde{f}|g,w_1,w_2,\dots w_n\right)
\end{aligned}$$

Hence $\lim h_f(gw_n) = \tilde{f}(g,w)$ hence $f - h_f$ indeed in $\mathscr{I}$.

■

Remark 9 This lemma can be consider as an if-and-only-if version of the martingale convergence theorem. Being bounded harmonic is not just a good reason for a.s. convergence but essentially the only reason for a bounded function to converge.

Corollary 5 $\mathscr{H}$ is not just a Banach space - there is a multiplication that makes it an algebra, with a G-action on it.

This multiplication is the pointwise multiplication modulo $\mathscr{I}$. That is, $h_1 \cdot h_2 = h$ where $h_1 h_2 = h + i$.

Explicitly,

Thus, $\mathscr{H}$ admits a spectrum Π_{top}: this is a compact space such that $\mathscr{H} \approx \mathscr{C}\left(\Pi_{top}\right)$ (Gelfand representation). For $h \in \mathscr{H}$, denote by $\bar{h}$ the image by Gelfand.

Π_{top} is a G-space

Since $\mathscr{H}$ is a G-space, we can push the G-action forward to $\mathscr{C}\left(\Pi_{top}\right)$. That is, any $g \in G$ defines an automorphism of $\mathscr{C}\left(\Pi_{top}\right)$. Such an automorphism always induced (Stone-Banach) by some homeomorphisms of Π_{top}. This defined a G-action on Π_{top}.

The measure ν

Until now we have a G-sapce. We are looking for a measure on the space.

Define for each $g \in G$, $L_g : \mathscr{H}^\infty \to \mathbb{R}$ by

$$L_g(h) = h(g) = \int_\Omega \tilde{h}(g,\omega)\, d\mathbb{P}(\omega).$$

In words, given h, and an element g integrate the limiting values of h along all the future conditioned that we start to walk from g.

It induces a positive linear function $\bar{L}_g$ on $\mathscr{C}(\Pi_{top})$ and by Riesz, we have a $\nu_g \in \mathscr{P}(\Pi_{top})$ such that for $\bar{h} \in \mathscr{C}(\Pi_{top})$,

$$\nu_g(\bar{h}) = \int_{\Pi_{top}} \bar{h}(x)\, d\nu_g(x) = \bar{L}_g(\bar{h}) = L_g(h) = \int_\Omega \tilde{h}(g,\omega)\, d\mathbb{P}(\omega) = h(g)$$

Properties of these measures:

- **The G-action is** $g\nu_e = \nu_g$
  $g\nu_e(\bar{h}) = \nu_e(\bar{h}^g) = \nu_e\left(\overline{h^g}\right) = h^g(e) = h(g) = \nu_g(\bar{h})$
- $\nu = \nu_e$ **is stationary.**
  $\nu(\bar{h}) = h(e) = \sum_{\gamma\in G} \mu(\gamma) h(\gamma) = \sum_{\gamma\in G} \mu(\gamma)\nu_\gamma(\bar{h}) = \mu * \nu(\bar{h})$
- **Gelfand=Poisson**
  With this measure, the Gelfand representation $h \leftrightarrow \bar{h}$ and the Poisson transform coincide: For $\bar{h}$ we need to show that $h = \mathscr{P}_\nu(\bar{h})$.

$$\mathscr{P}_\nu(\bar{h})(g) = g\nu(\bar{h}) = \nu_g(\bar{h}) = h(g)$$

- (Π_{top}, ν) **is a boundary.**
  Fix w such that ν_w exists, and let $\bar{h} \in \mathscr{C}(\Pi_{top})$.

$$\begin{aligned}\int_{\Pi_{top}} \bar{h}(x) d\nu_w(x) &= \lim_{n\to\infty} \int \bar{h}(x) d(w_n\nu)(x) \\ &= \lim_{n\to\infty} \bar{L}_{w_n}(\bar{h}) = \lim_{n\to\infty} L_{w_n}(h) \\ &= \lim_{n\to\infty} h(w_n)\end{aligned}$$

Apply the last equations on h^2 to get

$$\lim_{n\to\infty} h^2(w_n) = \int_{\Pi_{top}} \bar{h}^2(x) d\nu_w(x).$$

Recall that the Gelfand map $h \leftrightarrow \bar{h}$ is multiplicative. This is actually a tautology: we defined the multiplication on $\mathscr{H}$ by this map. Thus,

$$\lim_{n\to\infty} h^2(w_n) = (\lim_{n\to\infty} h(w_n))^2 = \left(\int_{\Pi_{top}} \bar{h}(x)\, d\nu_w(x)\right)^2$$

But from Cauchy-Schwartz we get

$$\int_{\Pi_{top}} \left(\bar{h}(x)\right)^2 d\nu_\omega(x) \geq \left(\int_{\Pi_{top}} \bar{h}(x)\, d\nu_\omega(x)\right)^2$$

So we have an equality in Cauchy-Schwartz and so $\bar{h}$ must be constant on Supp(ν_ω). But it holds for any $\bar{h} \in \mathscr{C}\left(\Pi_{top}\right)$ so ν_ω is a Dirac measure.

Universality and Uniqueness

Let (B,η) be a compact boundary. Then we showed that the Poisson transform

$$\mathscr{P}_\nu : L^\infty(B,\eta) \to H^\infty(G,\mu)$$

form an isometry. We now further claim:

Lemma 17 If (B,ν) is a compact boundary then $\mathscr{P}_\nu$ is multiplicative $\mathscr{P}_\nu : L^\infty(B,\nu) \to \mathscr{H}^\infty(G,\mu)$.

Proof. We need to show that for any $f_1, f_2 \in L^\infty(B,\nu)$, $\mathscr{P}_\nu(f_1 \cdot f_2) = \mathscr{P}_\nu(f_1) \cdot \mathscr{P}_\nu(f_2)$, where the first multiplication is the pointwise multiplication on X and the second is as the multiplication of harmonic functions (modulo $\mathscr{I}$).

So we need to show that for a.e. $w \in (\Omega, \mathbb{P}_\mu)$,

$$\lim_n \mathscr{P}_\nu(f_1 \cdot f_2)(w_n) - \mathscr{P}_\nu(f_1)(w_n) \cdot \mathscr{P}_\nu(f_2)(w_n) = 0.$$

Let w be such that ν_w exists and denote by $b \in B$ the point such that $\nu_w = \delta_b$.

$$\begin{aligned} \lim_n \int_B f_1(x) f_2(x) d w_n \nu(x) &= \int_B f_1(x) f_2(x) d\nu_w(x) \\ &= f_1(b) f_2(b) \end{aligned}$$

And since

$$\lim_n \int_B f_i(x) d w_n \nu(x) = f_i(b)$$

we conclude that $\mathscr{P}_\nu$ is multiplicative. ■

Corollary 6 For a compact μ-boundary, the image $Im(\mathscr{P}_\nu)$ is a subalgebra $\mathscr{S} \leq H^\infty(G,\mu)$. Moreover, the Poisson transform is a C^*-isomorphism $L^\infty(B,\nu) \approx \mathscr{S}$.

By correspondence between sub C^* algebras and compact factors, we get that B is continues factor of Π_{top}.

In particular, any bounded harmonic function is given as a Poisson transform of some function on the Furstenberg-Poisson boundary.

4.2.3 Disadvantage of the topological Poisson boundary

Up to here we constructed a compact topological stationary space Π_{top} such that every compact boundary is a continuous factor of Π_{top}. In general this space Π_{top} is too large. In particular, there is no reason that it will be metrizable, so it is unnatural to work with.

For example, in order to prove weak* converges (which plays an important role in this theory), one must show that for every continuous function the limit exists, when we ignore a zero measure set. Thus if $\mathscr{C}(X)$ is separable it is enough to show it for each function from the dense set (and the intersection of the correlated full measure sets will still be of full measure).

Moreover, for discrete groups, Π_{top} is not metrazible, although we know nice and intuitive models for the Poisson boundary of F_2 or lamplighter groups.

The deep point here is that factor (not continuous one) of a boundary is a boundary. Hence, instead of talking about "continuous-universal space" we can ask for just universal space.

That is, we can take a different compact model for Π_{top}. Any boundary will be a factor (again, not continuous anymore) of it, and then this space can be chosen to be separable (in this procedure one can take the advantage of G being separable). However, We won't describe this construction here, but we will consider a whole different prespective: We will construct an abstract space, and then one can take any compact model of it.

As topological space, neither of these compact models won't be unique in the topological category - and indeed for many groups, there are different (G-isomorphic but non homeomorphic) models for the Poisson boundary which are natural with respect to the group.

The advantage of this topological construction is the intuition: we understand what is the meaning of the measure ν - this is just the hitting distribution on the boundary. Moreover, for any $g \in G$ the measure $g\nu$ represents the hitting distribution for a random walk that starts from g. Thus the G-action can be though as a re-rooting.

4.3 Zimmer's Construction (shift ergodic components)

4.3.1 Arrows and stars

Let $(X,\mathscr{A},\nu) \xrightarrow{\pi} (Y,\mathscr{B},\eta)$.

Recall that the measure ν can be disintegrated $\nu = \int_Y \nu_y d\eta(y)$ with $supp(\nu_y) \subset \pi^{-1}(y)$

There two natural maps:

- $\pi^* : L^\infty(Y,\mathscr{B},\eta) \to L^\infty(X,\mathscr{A},\nu)$ which is composition: $(\pi^* f)(x) = f(\pi(x))$.

Note that

$$\nu(\pi^* f) = \int_X \pi^* f(x)\, d\nu(x) = \int_X f(\pi(x))\, d\nu(x) = \int_Y f(y)\, d\pi_*\nu(y) = \pi_*\nu(f)$$

This is the easy direction, and it works also as a function $\pi^* : \mathscr{C}(Y) \to \mathscr{C}(X)$.

- $\pi_* : L^\infty(X,\mathscr{A},\nu) \to L^\infty(Y,\mathscr{B},\eta)$. Given $f \in L^\infty(X,\mathscr{A},\nu)$, $(\pi_* f)(y) = \int f(x)\, d\nu_y(x) = \nu_y(f)$.

Claim 3 $\pi_*\pi^* = id_{L^\infty(Y)}$ and $\pi^*\pi_* = \mathbb{E}\left(\cdot|\pi^{-1}\mathscr{B}\right)$

Proof. Observe that

$$(\pi_*\pi^* f)(y) = \int_X (\pi^* f)(x)\, d\nu_y(x) = \int_X f(\pi(x))\, d\nu_y(x) = f(y)$$

where the last equality follows from the facts that $supp(\nu_y) \subset \pi^{-1}(y)$.

In the other direction, for $f \in L^\infty(X,\mathscr{A},\nu)$ we claim that $\pi^*\pi_* f = \mathbb{E}\left(f|\pi^{-1}\mathscr{B}\right)$.

It is clear that it is measurable in $\pi^{-1}\mathscr{B}$. And for any $B \in \mathscr{B}$,

$$\begin{aligned}
\int_{\pi^{-1}B} f(x)\,d\nu(x) &= \int_X 1_{\pi^{-1}B}(x)\cdot f(x)\,d\nu(x) \\
&= \int_Y \nu_y\left(1_{\pi^{-1}B}\cdot f\right)d\eta(y) \\
&= \int_B \nu_y(f)\,d\eta(y) \\
&= \int_B (\pi_* f)\,d\eta \\
&= \int_B (\pi_* f)\,d\pi_*\nu \\
&= \int_{\pi^{-1}B} \left(\pi^*\pi_* f\right)d\nu
\end{aligned}$$

So the function $\pi^*\pi_* f$ satisfies the properties of the conditional expectation, so by the uniqueness we get equality. ■

4.3.2 Construction by Shift ergodic components

Let $(\Omega,\mathscr{B},\mathbb{P}) = (G,\mu)^{\mathbb{N}}$. Define $T:\Omega\to\Omega$ by $T(\omega) = (\omega_1\cdot\omega_2,\omega_3,\dots)$ and consider the G action $g\omega = (g\omega_1,\omega_2,\omega_3,\dots)$.

These two actions, T and G commute:

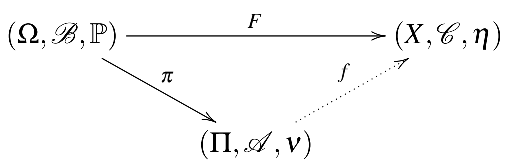
$$T(g\omega) = T(g\omega_1,\omega_2,\dots) = (g\cdot\omega_2,\omega_3,\dots) = g(\omega_2,\omega_3,\dots) = gT(\omega)$$

Now consider $\mathscr{I}_T$ to be the σ-algebra on Ω of T-invariant sets, that is $\mathscr{I}_T = \left\{B\in\mathscr{B} \mid T^{-1}B = B\right\}$.

The Mackey corresponding factor, $(\Pi,\nu) = T\backslash\backslash(\Omega,\mathbb{P})$ is the Poisson-Furstenberg boundary.

The following will give us characterization of (Π,ν), in terms of the measurable functions from it. It is a general fact about Mackey spaces, when we apply it to our case.

Lemma 18 Let (X,η) be some probability space. Every measurable $f:\Pi\to X$ defines a T-invariant function $F:\Omega\to X$. Moreover, every T-invariant $F:\Omega\to X$ is defined by some $f:\Pi\to X$.



Proof. Let $F:\Omega\to X$ be a measurable function. We want to show that F is T-invariant if and only if there exists such an f.

First we observe that

- F is T-invariant if and only if F is measurable w.r.t. $\mathscr{I}_T$: if $F\circ T = F$ then for any measurable $C\in\mathscr{C}$,
$$\begin{aligned}
F^{-1}(C) &= (F\circ T)^{-1}(C) \\
&= T^{-1}\left(F^{-1}(C)\right)
\end{aligned}$$

so $F^{-1}(C) \in \mathscr{I}_T$. For the other direction, assume F is $\mathscr{I}_T$-measurable we show that $F^{-1}(C) = (F \circ T)^{-1}(C)$ for any $C \in \mathscr{C}$. $(F \circ T)^{-1}(C) = T^{-1}\left(F^{-1}(C)\right)$ but F is $\mathscr{I}_T$-measurable so $F^{-1}(C) \in \mathscr{I}_T$ so indeed $T^{-1}\left(F^{-1}(C)\right) = F^{-1}(C)$.

- There exists f if and only if F is $\mathscr{I}_T$-measurable: if there exists such f, such that $F = \pi^* f$ so F is $\pi^{-1}\mathscr{A} = \mathscr{I}_T$ measurable, by definition. If F is $\mathscr{I}_T$-measurable then we can define $f = \pi_* F$. Now

$$\pi^* f = \pi^* \pi_* F = \mathbb{E}\left(F | \pi^{-1}\mathscr{A}\right) = \mathbb{E}(F | \mathscr{I}_T) = F.$$

■

Lemma 19 (Π, ν) is a stationary space.

Proof. Note that $(\Omega, \mathbb{P})$ is not a stationary space: $\mu * \mathbb{P} = \mu * \mu \times \mu \times \cdots \neq \mathbb{P}$: for $F \in L^\infty(\Omega, \mathbb{P})$,

$$\begin{aligned}
\int F(\omega_1, \omega_2, \omega_3 \ldots) \, dT_* \mathbb{P} &= \int F\left(T(\omega_1, \omega_2, \omega_3 \ldots)\right) d\mathbb{P} \\
&= \int F\left(\omega_1 \omega_2, \omega_{3,\ldots}\right) d\mathbb{P} \\
&= \int F d\left(\mu * \mu \times \mu^{\mathbb{N}}\right)
\end{aligned}$$

so we have $\mu * \mathbb{P} = T_* \mathbb{P}$. Since Π is the factor of T-invariant sets, both $\mathbb{P}$ and $T_* \mathbb{P}$ project to the same measure ν, so it is a stationary measure.

More formally, observe that if $F : \Omega \to X$ comes from a function on Π then it is T-invariant, hence $T_* \mathbb{P}(F) = \mathbb{P}(F)$. So for any $f \in L^\infty(\Pi, \nu)$ we have

$$\begin{aligned}
\mu * \nu(f) &= \mu * (\pi_* \mathbb{P})(f) = \pi_*(\mu * \mathbb{P})(f) = \mu * \mathbb{P}\left(\pi^* f\right) \\
&= T_* \mathbb{P}\left(\pi^* f\right) = \mathbb{P}\left(\pi^* f\right) = \pi_* \mathbb{P}(f) = \nu(f)
\end{aligned}$$

so $\mu * \nu = \nu$. ■

Lemma 20 The Furstenberg transform $F : L^\infty(\Pi, \nu) \to \mathscr{H}^\infty(G, \mu)$ is a bijection.

In other words: there is a bijection between T-invariant functions on Ω and G-harmonic functions.

Proof. Recall that for $f \in L^\infty(\Pi, \nu)$, $F(f)(g) = \int f dg\nu$ in this proof we write $\hat{f} = F(f)$.

On the other hand, for $h \in \mathscr{H}^\infty(G, \mu)$ recall that by the martingale convergence theorem the limit $\tilde{h}(e, \omega) = \lim_{n \to \infty} h(\omega_1 \cdots \omega_n)$ exists for a.e. $\omega \in \Omega$. That is, h defines some function on Ω which is T-invariant. Hence it defines a function $\bar{h} \in L^\infty(\Pi, \nu)$.

We now claim that these are inverse transforms of each other, that is $\bar{\hat{f}} = f$ and $\hat{\bar{h}} = h$:

Let $f \in L^\infty(\Pi, \nu)$ and denote by $\tilde{f} = \pi^* f \in L^\infty(\Omega, \mathbb{P})$ so $\tilde{f}$ is T-invariant. Note that $\hat{f}(g) = \int f dg\nu = \int \pi_* \tilde{f} dg \pi_* \mathbb{P} = \int \tilde{f} dg \mathbb{P}$. Denote $\bar{\omega} = \pi(\omega)$.

$$\begin{aligned}
\bar{\hat{f}}(\bar{\omega}) &= \lim_{n\to\infty}\hat{f}(\omega_1\cdots\omega_n) \\
&\lim_{n\to\infty}\int\limits_{\Omega}\tilde{f}d(\omega_1\cdots\omega_n\mathbb{P}) \\
&= \lim_{n\to\infty}\int\limits_{\Omega}\tilde{f}(\omega_1\cdots\omega_n,\omega')\,d\mathbb{P}(\omega') \\
&= \lim_{n\to\infty}\int\limits_{\Omega}\tilde{f}T^n(\omega_1,\ldots,\omega_n,\omega')\,d\mathbb{P}(\omega') \\
&= \lim_{n\to\infty}\int\limits_{\Omega}\tilde{f}(\omega_1,\ldots,\omega_n,\omega')\,d\mathbb{P}(\omega') \\
&= \tilde{f}(\omega) = f(\bar{\omega})
\end{aligned}$$

For $h \in \mathscr{H}^\infty(G,\mu)$,

$$\begin{aligned}
\hat{\bar{h}}(g) &= g\nu(\bar{h}) = g\pi_*\mathbb{P}(\bar{h}) \\
&= g\mathbb{P}\left(\lim_{n\to\infty}h(\omega_1\cdots\omega_n)\right) \\
&= \lim_{n\to\infty}\int\limits_{\Omega}h(\omega_1\cdots\omega_n)\,dg\mathbb{P}(\omega) \\
&= \lim_{n\to\infty}\int\limits_{\Omega}h(g\omega_1\cdots\omega_n)\,d\mathbb{P}(\omega) \\
&= \lim_{n\to\infty}\int\limits_{G}\cdots\int\limits_{G}h(g\omega_1\cdots\omega_n)\,d\mu(\omega_1)\cdots d\mu(\omega_n) \\
&= \lim_{n\to\infty}h(g) = h(g)
\end{aligned}$$

■

Corollary 7 The G-action on (Π,ν) is ergodic, that is every G-invariant set $A \subset B$ (that is $\nu(gA\Delta g) = 0$ for all $g \in G$) is of trivial measure ($\nu(A) \in \{0,1\}$)

Proof. Let A be a G-invariant set and define $f = \chi_A$. So f is G-invariant: $gf = g(\chi_A) = \chi_{g^{-1}A} = \chi_A = f$.

By the equivariance of the Furstenberg transorm, if $h \in \mathscr{H}^\infty$ is $F(f)$ then

$$h(g) = \int f\,dg\nu = \int gf\,d\nu = \int f\,d\nu = h(e)$$

so h is a constant function $h \equiv c$. Since constant functions on $L^\infty(\Pi,\nu)$ yield constant functions, by the injectivity of F we conclude that f is constant, that is, $\nu(A) \in \{0,1\}$. ■

4.3.3 Compact and abstract boundaries

We originally defined a boundary as a compact stationary space with Dirac measures as limiting measures.

Then we say that being a boundary is equivalent to being a factor of the Poisson boundary.

As such we can think of boundaries as G-invariant sub σ-algebras of the Poisson boundary.

The Furstenberg transform of a boundary is an isomorphism onto a G-invariant sub algebra of the algebra of $\mathscr{H}^{\infty}(G,\mu)$, and actually any such sub algebra corresponded to some boundary.

Theorem 7 Let (B,η) be an abstract (G,μ)-stationary space. Then the following are equivalent:
1. (B,η) is a factor of the Poisson boundary
2. Any compact model of (B,η) is a compact (G,μ)-boundary.
3. There exists a compact model of (B,η) which is a compact (G,μ)-boundary.

In this case we call (B,η) an abstract (G,μ)-boundary (or just boundary).

Proof. $(1\Rightarrow 2)$ Let $\pi:(\Pi,\nu)\to(B,\eta)$, and consider B as a compact metric space. Find $C_0\subset\mathscr{C}(B)$ a countable dense set and let $D=\{\pi^*f\}_{f\in C_0}\subset\mathscr{C}(\Pi)$.

For any $\pi^*f\in L^{\infty}(\Pi,\nu)$, $\overline{\widehat{\pi^*f}}=\pi^*f$. For any $f\in C_0$ there exists full measure Ω_f such that the equality holds for any $\bar{\omega}\in C_f$ (where $\bar{\omega}$ is the projection from $\Omega\to B$). Intersect all these sets to get a full measure $\Omega'\subset\Omega$ such that the equality holds for all $\omega\in\Omega'$ and $f\in C_0$.

We get that for any $\bar{\omega}\in\Omega'$ and any $\pi^*f\in D$ we have

$$\pi^*f(\bar{\omega})=\overline{\widehat{\pi^*f}}(\bar{\omega})=\lim_{n\to\infty}\widehat{\pi^*f}(\omega_1\cdots\omega_n)=\lim_{n\to\infty}\int\pi^*f\,d(\omega_1\cdots\omega_n)\,\nu$$

On the other hand, $\int\pi^*f\,d\omega_1\cdots\omega_n\nu=\int f\,d\omega_1\cdots\omega_n\eta$ so we get $\omega_1\cdots\omega_n\eta(f)\to\pi^*(f)(\bar{\omega})=f(\pi(\bar{\omega}))$ for any $f\in C_0$ and any $\omega\in\Omega'$. By continuity, this convergence holds for any $f\in\mathscr{C}(B)$ so by definition, $\omega_1\cdots\omega_n\eta\to\delta_{\pi(\bar{\omega})}$ for almost every $\omega\in\Omega'$.

$(2\Rightarrow 3)$ is clear.

$(3\Rightarrow 1)$ We consider now a compact boundary (B,η). The function $\omega\mapsto\eta_\omega=\delta_b$ is a measurable function $\Omega\to B$. This map is clearly T-invariant, hence it defines a function $\pi:\Pi\to B$. ■

Let μ be the SRW on $\mathbb{F}_2$. We claim that there is a stationary measure on the boundary of the tree which is μ-boundary.

For any r, let $\tau_r:G^{\mathbb{N}}\to\mathbb{N}$ be such that $\tau_r(w)=\max_n\{\|g_1\cdots g_n\|\le r\}$ Each τ_r is defined on a full measure set, and consider its intersection. Now define $bnd:\Omega\to\partial Tree$ by $bnd(w)_r=g_{\tau_r(w)}$.

Lamplighter group $\oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})\rtimes\mathbb{Z}$. Elements are of the form (f,k) where $f:\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}$ with finite support and $k\in\mathbb{Z}$. $\mathbb{Z}$ acts on $\oplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ by shifting the index. Multiplication is given by

$$(f_1,k_1)(f_2,k_2)=(f_1+k_1f_2,k_1+k_2)$$

generated by 3 elements. It follows that

$$(f,k)^{-1}=(-kf,-k)$$

here we used that $f^{-1}=f$.

The commutator subgroup is the lamps, and the quotient is $\mathbb{Z}$ so it is amenable (solvable).

Consider the space Ω and denote by $w_n=(f_n,k_n)$. Project μ to $\bar{\mu}$ on $\mathbb{Z}$. If $\bar{\mu}$ is not symmetric then there is a map $conf:\Omega\to\Pi(\mathbb{Z}/2\mathbb{Z})$ which is given by $conf(w)(z)=\lim f_n(z)$. This map is shift invariant so we get a μ-boundary.

5. Properties of the Furstenberg-Poisson boundary

Corollary 8 Since we know that the Poisson boundary of finite index subgroup with the hitting measure is the same as the original we get that for any vitrually nilpotent group any generating measure, the Poisson boundary is trivial.

Open problem: It is an open question to describe the class of groups that have trivial Poisson boundary for any measure.

5.1 Operations of group, and the Poisson boundary

No endomorphism: Barycenter map: Let M be a convex space, and consider $\text{Prob}(M)$. As always, we can embed M in $\text{Prob}(M)$ via $m \mapsto \delta_m$.

Since M is convex, we have also a map in the other direction, called the barycenter map. $bar : \text{Prob}(M) \to M$ which is the continues extension of the map $bar(\sum_{i=1}^n p_i \delta_{m_i}) = \sum_{i=1}^n p_i m_i$. The composition $M \to \text{Prob}(M) \to M$ is the identity on M.

The canonical example is when $M = \text{Prob}(X)$. When G acts on X, we get an action on M and the map bar is equivariant.

Lemma 21 Let (X, ν) be a μ-boundary. Then the only endomorphism is the identity. Namely, if $\alpha : X \to X$ is a measurable map that commutes with the G-action, then $\alpha = id$.

Proof. Let $\eta = \frac{1}{2}(\nu + \alpha_* \nu)$. So η is a μ-stationary and it's conditional measures are given by

$$\eta_w = \frac{1}{2}(\nu_w + \alpha_* \nu_w) = \frac{1}{2}(\delta_{\mathbf{bnd}(w)} + \delta_{\alpha(\mathbf{bnd}(w))}).$$

We claim that η is a μ-boundary, which implies that $\alpha = id$. Define $\tilde{\alpha} : X \to \text{Prob}(X)$ by $\tilde{\alpha}(x) = \frac{1}{2}(\delta_x + \delta_{\alpha(x)})$.

The ν pushforward via $\text{Prob}(X) \xrightarrow{\tilde{\alpha}_*} \text{Prob}(\text{Prob}(X))$ given by $\frac{1}{2}(\delta_\nu + \delta_{\alpha_* \nu})$.

Now we can use $bar : \text{Prob}(\text{Prob}(X)) \to \text{Prob}(X)$ to push this measure to get $bar_*(\tilde{\alpha}_* \nu) = \frac{1}{2}(\nu + \alpha_* \nu) = \eta$.

Hence η is a μ-boundary as an equivariant image of the μ-boundary ν. ■

5.1.1 Quotient groups

Let (G,μ) and let $G \to Q$ with kernel N. Denote by $\bar{\mu} \in \mathscr{P}(Q)$ to be the projected random walk. It is just the push-forward $\varphi_*\mu$ where φ is the group homomorphism.

Let (X,ν) be a $(Q,\bar{\mu})$-stationary space. Then we can define a G-action on X that factorizes through φ, that is $gx = \varphi(g)x$.

Claim 4 (X,ν) is (G,μ)-stationary if and only if it is $(Q,\bar{\mu})$-stationary.
Moreover, ν is μ-boundary if and only if it is $\bar{\mu}$-boundary.

Proof. $\sum_{g\in G}\mu(g)g\nu = \sum_{g\in G}\mu(g)\varphi(g)\nu = \sum_{q\in Q}\bar{\mu}(q)q\nu$.
Now $\nu_w = \lim w_n\nu = \lim \varphi(w_n)\nu$ If (B,ν) is a $(Q,\bar{\mu})$-compact boundary then it is (G,μ)-compact boundary: which is a Dirac measure for a.e. $\bar{\mu}$-random walk. ■

Lemma 22 The Poisson boundary $\Pi(Q,\bar{\mu}) \approx N\backslash\backslash\Pi(G,\mu)$.

Proof. The space $N\backslash\backslash\Pi(G,\mu)$ is the maximal factor of $\Pi(G,\mu)$ on which N acts trivially. By the above, $\Pi(Q,\bar{\mu})$, is a μ-boudnary, that is, a factor of $\Pi(G,\mu)$ on which N acts trivially. Hence we get that $N\backslash\backslash\Pi(G,\mu) \to \Pi(Q,\bar{\mu})$.

On the other hand, any G-space on which N acts trivially, can be considered as a Q-space (pick elements g_q from $\varphi^{-1}(q)$). Hence we get a Q-action on $N\backslash\backslash\Pi(G,\mu)$. Once both Q and G act on $N\backslash\backslash\Pi(G,\mu)$, we get that it is $(Q,\bar{\mu})$-boundary, hence $\Pi(Q,\bar{\mu}) \to N\backslash\backslash\Pi(G,\mu)$.

The composition of these two maps is an endomorphism hence these spaces are isomorphic.
■

A different approach: Consider the two action on $(\Omega,\mathbb{P})$, now $\mathbb{P}$ is the Markov measures, so it holds the positions of the random walk and not the increments. One action is the shift, and the other is modding out each coordinate by N. These actions commute, and so we get that $\Pi(Q,\bar{\mu}) = N\backslash\Pi(G,\mu)$.

5.1.2 Recurrent subgroups

One interesting question is to relate the Poisson boundary of a group and its subgroups. In particular, if Γ is a "large" subgroup is G, can we identify the Poisson boundary of Γ with the one of G?

Remark 10 The historical motivation is coming from the study of Lie groups. A subgroup $\Gamma \leq G$ is a lattice if Γ is a discrete group and the space of cosets G/Γ admits a G-invariant measure. The first example is $\mathbb{Z} \leq \mathbb{R}$.

Is many senses lattices are "approximations" of the enveloping groups. One aspect, for example is the following question: can a lattice in $SL_2(\mathbb{R})$ be also a lattice in $SL_3(\mathbb{R})$. Nowadays, one can prove that it cannot be using many tools. However, the first proof was given by Furstenberg and it was the question that led him to define the Poisson boundary.

The philosophical idea that if one is very far, he cannot distinguish between the lattice and the enveloping group. The Poisson boundary is a way to consider groups from a far point of view - any local behavior of the random walk disappears in the Poisson boundary.

Let $\Gamma \leq G$ be a subgroup and let $\mu \in \mathscr{P}(G)$ be a random walk. Denote $(\Omega,\mathbb{P}) = (G,\mu)^{\mathbb{N}}$.

Definition 17 Γ is called μ-recurrent if the μ-random walk almost surely visits Γ infinitely many times.

■ **Example 5.1** $F_2 \times \mathbb{Z}_2$.

■ **Example 5.2** Let $\Gamma \leq F_2$ be the commutator group, that is, the group generated by elements of the form $[w_1, w_2] = w_1 w_2 w_1^{-1} w_2^{-1}$. Algebraically, Γ is isomorphic to the free group F_∞ with finitely many generators. The quotient F_2/Γ is isomorphic to $\mathbb{Z}^2$ and the projected random walk $\bar{\mu}$ is the simple random walk on $\mathbb{Z}^2$. Since $\mathbb{Z}^2$ is recurrent w.r.t. the simple random walk we conclude that Γ is μ-recurrent.

Definition 18 Let Γ be a μ-recurrent subgroup. Denote by $\tau : \Omega \to \mathbb{N}$, the first return time

$$\tau(\omega) = \min \left\{ n \geq 1 | \omega_1 \cdots \omega_n \in \Gamma \right\}.$$

Γ is μ-positive recurrent if $\mathbb{E}(\tau) < \infty$ and μ-null recurrent if $\mathbb{E}(\tau) = \infty$.

■ **Example 5.3 — Two floors F_2.** Let $G = F_2 \times \mathbb{Z}_2$ and let $\mu(e,1) = p$ and $\mu(g,0) = \frac{1-p}{4}$ for $g \in \left\{ a, a^{-1}, b, b^{-2} \right\}$. The subgroup $F_2 \times \{0\}$ is a μ-positive recurrent subgroup.

What about the commutator group is F_2?

Theorem 8 — Kac's formula. If Γ is μ-recurrent then $\mathbb{E}(\tau) = [G : \Gamma]$.

Proof. Consider the Markov chain where the states are $\Gamma \backslash G$ and perform the right random walk. That is, $X_0 = \Gamma$, the trivial coset and $X_n = \Gamma Z_n$ where $Z_n \sim \mu^n$. Observe that the original random walk visits Γ exactly when the Markov chain visits the trivial coset. This Markov chain is irreducible (due to the generating assumption), doubly stochastic and recurrent.

Now, if the index is finite, so the Markov chain is finite and there exists a stationary probability distribution η which is uniform. Thus by Kac's formula, the expected return time to Γe is $1/\eta(\Gamma e) = [G : \Gamma]$.

If the index is infinite, this Markov chain admits no finite stationary measure, and hence, it is null recurrent. In other words, $\mathbb{E}(\tau) = \infty$. ■

Corollary 9 The property μ-positive recurrent is independence on μ!

Definition 19 Let $\Phi : \Omega \to \Gamma$ be the **hitting map**:

$$\Phi(\omega) = \omega_1 \cdots \omega_{\tau(\omega)}.$$

Now we can push forward $\mathbb{P}$ to get a measure on Γ. Let $\theta \in \mathscr{P}(\Gamma)$ be $\theta = \Phi_* \mathbb{P}$.

■ **Example 5.4** Consider the two floors free group. θ has full support.

Definition 20 Now let (X, ν) be a (G, μ)-stationary space. We can consider the restricted action $\Gamma \curvearrowright X$. We will show that actually ν is a θ-stationary measure.

The proof relies on the optional stopping theorem from probability theory. We won't get into the details here and just rephrase it for our purposes.

Let Ψ be the Γ-restriction map. That is, for any $f : G \to \mathbb{R}$, we let $\Psi(f) : \Gamma \to \mathbb{R}$ defined by $\Psi(f)(\gamma) = f(\gamma)$.

Theorem 9 — Optional stopping theorem. The map $\Psi : \mathscr{H}^\infty(G, \mu) \to \mathscr{H}^\infty(\Gamma, \theta)$ is an isometric bijection.

First, one should prove that the restriction of a μ-harmonic function is θ-harmonic. Next, the interesting part is to observe that μ-harmonic functions are determined by their values on Γ.

Theorem 10 (X,ν) is (Γ,θ)-stationary. Moreover, (G,μ) and (Γ,θ) share the same Poisson boundary.

Proof. Let F_μ be the Furstenberg transform $F_\mu : L^\infty(X,\nu) \to \mathscr{H}^\infty(G,\mu)$ defined by $F_\mu(f)(g) = g\nu(f)$. This map can be defined for non stationary actions as well. Note that ν is μ-stationary if and only if the image contains only μ-harmonic functions.

Then $F_\theta = \Psi \circ F_\mu$, the Furstenberg transform for the (Γ,θ)-action, maps $L^\infty(X,\nu) \to \mathscr{H}^\infty(\Gamma,\theta)$. Since the image is inside the space of harmonic functions, ν is θ-stationary.

It follows that for $(X,\nu) = \Pi(G,\mu)$, $F_\theta = \Psi \circ F_\mu$ is an isometric bijection between $L^\infty(X,\nu)$ and $\mathscr{H}^\infty(\Gamma,\theta)$ which implies that (X,ν) is actually the Poisson boundary of (Γ,θ). ■

5.2 Relatively measure preserving factors

Let $\pi : (X,\nu) \to (Y,\eta)$. Consider the disintegration $\nu = \int \nu_y d\eta(y)$.

Definition 5.2.1 A factor map $\pi : (X,\nu) \to (Y,\eta)$ is relatively measure preserving if the disintegration map $X \to \mathrm{Prob}(Y)$ is equivariant.

Fact 1 $\pi : (X,\nu) \to \{*\}$ is relatively measure preserving if and only if (X,ν) is measure preserving.

Explicitly it means that if $g\nu_y = \nu_{gy}$ for every g and η-a.e. $y \in Y$, which is equivalent to $\frac{dg\nu_{g^{-1}y}}{d\nu_y}(x) = 1$ for ν_y-a.e. x and η-a.e. $y \in Y$ and every $g \in G$.

Lemma 23 The factor $\pi : (X,\nu) \to (Y,\eta)$ is relatively measure preserving if and only if the functions $\frac{dg\nu}{d\nu}$ are Y-measurable, for all g.

Remark 11 We may write these conditions just for $g \in supp(\mu)$. It is equivalent as for any $g \in G$ since we consider μ to be a generating measure.

Proof. Write the disintegrations:$\nu = \int \nu_y d\eta(y)$ and $g\nu$ w.r.t. $g\eta$:

$$g\nu = \int (g\nu)_y \, dg\eta(y) = \int (g\nu)_{gy} \, d\eta.$$

By the linearity of the G action on $\mathscr{P}(X)$ we get that

$$g \cdot \nu = g\left(\int \nu_y d\eta\right) = \int g \cdot \nu_y d\eta.$$

By the uniqueness of the disintegration, we get that $g\nu_y = (g\nu)_{gy}$.

Now the factor is relatively measure preserving is equivalent to $g\nu_y = \nu_{gy}$ if and only if $\nu_{gy} = (g\nu)_{gy}$, which we can rewrite as $\frac{d\nu_{gy}}{d(g\nu)_{gy}}(x) = 1$ for a.e. x.

$(4 \implies 3)$ Assume that $\frac{dg\nu}{d\nu}$ is Y-measurable so $\pi^* \frac{dg\eta}{d\eta} = \frac{dg\nu}{d\nu}$.

Now

$$\begin{aligned} \int f dg\nu &= \int f \frac{dg\nu}{d\nu} d\nu \\ &= \int \int \left(f \frac{dg\nu}{d\nu} \right) d\nu_y d\eta \end{aligned}$$

But the assumption is that $\frac{dg\nu}{d\nu}$ is constant on the fibers so

$$= \int \int f d\nu_y \pi_* \frac{dg\nu}{d\nu} d\eta = \int \int f d\nu_y \frac{dg\eta}{d\eta} d\eta.$$

Since it holds for any f, we have equality of measures:

$$g\nu = \int \nu_y \frac{dg\eta}{d\eta} d\eta$$

If we just disintegrate $g\nu$ we get

$$g\nu_y = (g\nu)_{gy}.$$

$$g\nu = \int (g\nu)_y \, dg\eta = \int \frac{(g\nu)_y}{dg\nu}$$

(4 $\iff$ 5) We already saw that in general $\varphi_Y \le \varphi_X$ and equality holds if and only if all the Radon-Nikudym derivatives in X are Y-measurable. ■

Consider the Furstenberg transforms from X and from Y (F_X, F_Y respectively).

Corollary 10 For any $\pi : (X, \nu) \to (Y, \eta)$, $Im(F_Y) \subset Im(F_X)$. If π is measure preserving then $Im(F_X) = Im(F_Y)$.

Proof. Note that the Furstenberg transform F_Y generates harmonic function using the functions $\frac{dg\eta}{d\eta}$:

$$F_Y(f)(g) = g\eta(f) = \int f dg\eta = \int f \frac{dg\eta}{d\eta} d\eta.$$

We can think of the functions $\frac{dg\eta}{d\eta}$ as a kernel for this transform: some kind of basis that using it F_Y generates bounded harmonic functions. Thus, since we always can pull back $\frac{dg\eta}{d\eta}$ to $\frac{dg\nu}{d\nu}$ it is clear that $Im(F_Y) \subset Im(F_X)$.

If π is measure preserving then all the "basis" that F_X uses, are actually already in Y so they share the same image. ■

Exercise 5.1 Prove the corollary without using RanonNikudym derivatives. For the equality part, use the definition of relatively measure preserving ($g\nu_y = \nu_{gy}$)

5.2.1 Product groups

Let us just mention now how the Poisson boundary behaves with products. We will give the proof later on, when after developing some machinery.

Let (G_1, μ_1) and (G_2, μ_2) be two random walks on two different group. Then

Theorem 11 $\Pi(G_1 \times G_2, \mu_1 \times \mu_2) = \Pi(G_1, \mu_1) \times \Pi(G_2, \mu_2)$.

Note that G_1 is a quotient group, and the projected measure is μ_1. Hence $\Pi(G_1, \mu_1)$ is a μ-boudnary. The same for (G_2, μ_2). Hence we get that $\Pi(G_1, \mu_1) \times \Pi(G_2, \mu_2)$ is a μ-boundary.

To claim that they are actually isomorphic we need the following notion.

We will discuss this notion later, but meanwhile, note that (X, ν) is measure-preserving if on only if $\Pi : (X, \nu) \to \{*\}$ is relatively measure preserving.

Lemma 24 If (X, ν) is a μ-boundary and $\pi(X, \nu) \to (Y, \eta)$ is relatively measure preserving then π is an isomorphism.

Proof. $X \to Y \to \text{Prob}(X) \to X$ is equivariant hence should be the identity. ■

6. Entropy theory

6.1 Furstenberg entropy

Let (X,ν) be a (G,μ)-stationary space.

Definition 21 The Radon-Nikudym cocycle $\rho : G\times X \to \mathbb{R}$ is

$$\rho(g,x) = -\log \frac{dg^{-1}\nu}{d\nu}(x)$$

This is an additive cocycle:

$$\begin{aligned}
\rho(g_1g_2,x) &= -\log \frac{dg_2^{-1}g_1^{-1}\nu}{d\nu}(x) \\
&= -\log\left(\frac{dg_2^{-1}g_1^{-1}\nu}{dg_2^{-1}\nu}(x)\,\frac{dg_2^{-1}\nu}{d\nu}(x)\right) \\
&= -\log\frac{dg_1^{-1}\nu}{d\nu}(g_2x) - \log\frac{dg_2^{-1}\nu}{d\nu}(x) \\
&= \rho(g_1,g_2x)+\rho(g_2,x)
\end{aligned}$$

by the chain rule of the Radon-Nikudym derivative.

Definition 22 Define $\varphi : G\to\mathbb{R}$ by

$$\varphi(g) = \int_X \rho(g,x)\,d\nu(x)$$

φ measures the deformation each g makes on ν. Indeed, using the convexity of $-\log$, by Jensen φ is non-negative:

$$\begin{aligned}\varphi(g) &= \int_X -\log\frac{dg^{-1}\nu}{d\nu}(x)\,d\nu(x)\\ &\geq -\log\int_X \frac{dg^{-1}\nu}{d\nu}(x)\,d\nu(x)\\ &= -\log\int_X dg^{-1}\nu(x)=0.\end{aligned}$$

Equality holds if and only if $\frac{dg^{-1}\nu}{d\nu}$ is constant a.e. that is, ν and $g^{-1}\nu$ are the same.

In terms of information theory, $\varphi(g)=D_{KL}(g\nu\|\nu)$ is the Kullback-Leibler divegence.

Definition 23 The **Furstenberg entropy** of a (G,μ) stationary space, (X,ν) is

$$h_\mu(X,\nu)=\int\varphi(g)\,d\mu(g).$$

This quantify the average deformation that G makes when it acts on ν. In particular, it follows for the discussion above that $h_\mu(X,\nu)=0$ if and only if ν is G-invariant (we also say in that case that (X,ν) is measure preserving).

Lemma 25 The entropy decreases with factors: if $\pi:(X,\nu)\to(Y,\eta)$ then $h_\mu(X,\nu)\geq h_\mu(Y,\eta)$.

Proof. Consider the conditional expectation operator $\pi_*:L^\infty(X,\nu)\to L^\infty(Y,\eta)$. Let $f>0$ with $\log f\in L^1(X,\nu)$. By Jensen inequality for conditional expectations, since $-\log$ is strictly convex we get

$$-\log\left(\pi_*(f)(y)\right)\leq\pi_*(-\log f)(y)$$

for η-a.e. $y\in Y$ and equality holds if and only if f is Y-measurable.

Apply it to $f=\frac{dg^{-1}\nu}{\nu}$. A general fact about Radon-Nikudym derivatives is that the projection of a derivative is the derivative of the projected measures.

In our case,

$$\pi_*\frac{dg^{-1}\nu}{d\nu}=\frac{d\pi_*\left(g^{-1}\nu\right)}{d\pi_*\nu}=\frac{dg^{-1}\eta}{d\eta}$$

So the Jensen inequality says that

$$-\log\frac{dg^{-1}\eta}{d\eta}(y)\leq\pi_*\left(-\log\frac{dg^{-1}\nu}{d\nu}\right)(y)$$

After integrating against η we get

$$\varphi_Y(g)\leq\int\pi_*\left(-\log\frac{dg^{-1}\nu}{d\nu}\right)(y)\,d\pi_*\nu=\int-\log\frac{dg^{-1}\nu}{d\nu}(x)\,d\nu(x)=\varphi_X(g)$$

In particular $h_\mu(Y,\eta)\leq h_\mu(X,\nu)$, and equality hold if and only if each of the $\frac{dg^{-1}\nu}{d\nu}$ are Y-measurable. ■

Lemma $h_{\mu^n}(X,\nu)=n\cdot h_\mu(X,\nu)$

Proof. **Exercise.** ■

6.2 Random walk entropy

Recall the definition of usual entropy of an atomic measure:

$$H(\mu) = \sum_{g \in G} -\mu(g) \log \mu(g).$$

with the convention that $0 \log(0) = 0$.

$$H(\mu) = \log(|X|) - D_{KL}(\mu||u)$$

In this section we will assume that $H(\mu) < \infty$.

Simple information theoretical argument shows that given two measures on the group, μ and μ',

$$H(\mu * \mu') \leq H(\mu) + H(\mu').$$

Thus, the sequence $H(\mu^n)$ is sub additive so the limit

$$h_{RW}(G, \mu) = \lim \frac{1}{n} H(\mu^n)$$

exists. This value (can be either 0 or ∞) called the **random walk entropy** (or **Avez entropy**, or asymptotic entropy).

Theorem 12 $h_\mu(X, \nu) \leq h_{RW}(G, \mu)$

Proof. $\nu = \mu * \nu = \sum_g \mu(g) \cdot g\nu$ then

$$1 = \frac{d\nu}{d\nu}(x) = \frac{d\left(\sum \mu(g) \cdot g\nu\right)}{d\nu}(x) = \sum \mu(g) \frac{dg\nu}{d\nu}(x) \geq \mu(g_0) \frac{dg_0\nu}{d\nu}(x)$$

so $\frac{dg\nu}{d\nu}(x) \leq \frac{1}{\mu(g)}$. Now, using the cocycle property,

$$\rho(g, x) = -\rho\left(g^{-1}, gx\right) = \log \frac{dg\nu}{d\nu}(gx) \leq -\log \mu(g)$$

So

$$h_\mu(X, \nu) = \sum_g \mu(g) \int_X \rho(g, x) d\nu(x) \leq \sum_g \mu(g)(-\log \mu(g)) = H(\mu).$$

Thus we get

$$h_\mu(X, \nu) = \frac{1}{n} h_{\mu^n}(X, \nu) \leq \frac{1}{n} H(\mu^n) \to h_{RW}(G, \mu)$$

■

Corollary 11 If $h_{RW}(G, \mu) = 0$ then the Poisson boundary is trivial.

Definition Let G be a finitely generated group and fix a generating set S.

G has **exponential growth** if $\lim_{n\to\infty}|S^n|^{\frac{1}{n}} > 1$ and subexponential otherwise.

G is said to have **polynomial growth** if $\lim_{n\to\infty}\frac{|S^n|}{n^d} \leq C$ for some fixed C and d.

One can prove that these properties are group properties - that is, independent of choice of S.

Remark 12 A highly non-trivial theorem of Gromov states that G has polynomial growth if and only if it is virtually nilpotent.

It was open question whether there are intermediate growth groups - subexponential groups which growth faster then any polynomial.

The first such groups was constructed by Grigichuk.

Corollary 12 If G has subexponential growth then for any generating μ with finite support then $\Pi(G,\mu)$ is trivial.

Proof. $H(\mu^n) \leq \log|S^n|$ and then $\frac{1}{n}H(\mu^n) \leq \frac{1}{n}\log|S^n| = \log|S^n|^{\frac{1}{n}} \to 0$. ■

Note that for polynomial groups, we know much more - by Gromov - it is virtually nilpotent and so for **any** measure the Poisson boundary is trivial (any generating measure - with no further assumptions like finite support or finite moments). However, it applies also for finitely supported measures, on intermediate growth.

6.2.1 Equality of the entropies

Theorem 13 [Kaimanovich-Vershik] For $h_{RW}(G,\mu) = h_\mu(\Pi,\nu)$.

Let $(\Omega,\mathbb{P}_\mu)$, where $\mathbb{P}_\mu$ is as usual the Markov measure and not the product.

Consider the following partitions of the space. α_1 is the partitions by the first cylinder, and denote by $\alpha_1(\omega)$ the element in the partition α_1 that contains ω. Let τ_n to be the partition where $\omega \sim_{\tau_n} \omega'$ if they agree after time n. Note that $\{\tau_n\}$ is a decreasing family of sigma algebras, and the tail sigma algebra is the limit $\tau = \lim_{n\to\infty}\tau_n$.

Another construction of the Poisson boundary was given by Kaimanovich-Vershik when they realized the Poisson boundary as the space $(\Omega,\mathbb{P}_\mu)$ when equipped with the tail sigma algebra. It is very similar to Zimmer's construction where he considered the sigma algebra of invariant sets. The invariant sigma algebra and the tail sigma algebra are different.

$w \sim_I w'$ if there exists some N such that for all $n < N$, $w_n = w'_n$.

$w \sim_\tau w'$ if there exist some k and N such that for all $n > N$, $w_{n+k} = w'_n$.

However, for generating μ, these two sigma algebras are equivalent mod $\mathbb{P}_\mu$. So let's think of Poisson boundary as the realization of the tail sigma algebra. In that case, $\mathbf{bnd} : (\Omega,\mathbb{P}_\mu) \to \Pi(G,\mu)$ is given by $w \mapsto \tau(w)$.

Given a space, say $(\Omega,\mathbb{P})$ and a sigma algebras, say α_1, the entropy if α_1 is defined to be

$$H(\alpha_1) = \int_\Omega -\log\mathbb{P}\left(\alpha_1(\omega)\right)d\mathbb{P}(\omega)$$

which coincide, in this case, with $H(\mu)$.

Given another sigma algebra, say τ, the conditional entropy of α_1 w.r.t. τ is

$$H(\alpha_1|\tau) = \int_\Omega -\log\mathbb{P}\left[\alpha_1(\omega)|\tau(\omega)\right]d\mathbb{P}(\omega).$$

We will prove the theorem by showing that the both quantities are equal to

$$h_{RW}(G,\mu) = H(\mu) - H(\alpha_1|\tau) = h_\mu(\Pi,\nu).$$

Lemma 26 $h_{RW}(G,\mu) = H(\mu) - H(\alpha_1|\tau)$.

Proof. Let $\omega \in \Omega$. write $z_n = \omega_1 \cdots \omega_n$. Then,

$$\mathbb{P}(\alpha_1(\omega)|\tau_n(\omega)) = \frac{\mu(z_1)\cdot\mu^{n-1}\left(z_1^{-1}z_n\right)}{\mu^n(z_n)}.$$

and so $H(\alpha_1|\tau_n) = \int\limits_{\Omega} -\log\mathbb{P}(\alpha_1(\omega)|\tau_n(\omega))\,d\mathbb{P}(\omega) = H(\mu) + H\left(\mu^{n-1}\right) - H(\mu^n)$.

$H(\alpha_1|\tau_n) \leq H(\alpha_1|\tau_{n+1})$ since τ_{n+1} is a sub partition of τ_n, so knowing τ_n gives at least the amount of information about the first step, as τ_{n+1}. Thus,

$$H(\mu) + H\left(\mu^{n-1}\right) - H(\mu^n) \leq H(\mu) + H(\mu^n) - H\left(\mu^{n+1}\right)$$

and so

$$H\left(\mu^{n-1}\right) - H(\mu^n) \leq H(\mu^n) - H\left(\mu^{n+1}\right).$$

That is, the sequence $F_n = H(\mu^n) - H\left(\mu^{n-1}\right)$ is a decreasing sequence of non-negative numbers and hence it convergences, and so also its Cesaro sums. Now

$$h_{RW}(G,\mu) = \lim \frac{1}{n} H(\mu^n) = \lim \frac{1}{n}\sum_{k=1}^{n} F_k = \lim F_n.$$

Therefore,

$$H(\alpha_1|\tau) = \lim_{n\to\infty} H(\alpha_1|\tau_n) = H(\mu) - \lim_{n\to\infty}\left(H\left(\mu^{n-1}\right) - H(\mu^n)\right) = H(\mu) - h_{RW}(G,\mu).$$

■

Lemma 27 $h_\mu(\Pi,\nu) = H(\mu) - H(\alpha_1|\tau)$.

Proof. Let $E \subset \Pi$ with $\nu(E) > 0$ and $[g] \subset \Omega$ be a cylinder. Then

$$\mathbb{P}\left([g] \cap \text{bnd}^{-1}E\right) = \mathbb{P}([g])\,\mathbb{P}_g\left(\text{bnd}^{-1}E\right) = \mathbb{P}([g])\,g\nu(E).$$

And so

$$\mathbb{P}\left([g]\,|\text{bnd}^{-1}E\right) = \mathbb{P}([g])\,\frac{g\nu(E)}{\nu(E)}.$$

Since it holds for any E, we can write the density of this measure for a single $b \in \Pi$ (for a.e. b):

$$\mathbb{P}\left[\alpha_1(w)|\tau(\omega)\right] = \mu(w_1)\frac{dw_1\nu}{d\nu}(\text{bnd}(w))$$

$$\begin{aligned} H(\mu) - H(\alpha_1|\tau) &= -\sum_g \mu(g)\log\mu(g) + \int_\Omega \log\mathbb{P}\left[\alpha_1(\omega)|\tau(\omega)\right] d\mathbb{P}(\omega) \\ &= -\sum_{w_1}\mu(w_1)\log\mu(w_1) + \sum_{w_1}\mu(w_1)\int_\Omega \log\mathbb{P}\left[\alpha_1(\omega)|\tau(w)\right] d\mathbb{P}_{w_1}(w) \\ &= -\sum_{w_1}\mu(w_1)\log\mu(w_1) + \sum_{w_1}\mu(w_1)\int_\Omega \log\left(\mu(w_1)\frac{dw_1\nu}{d\nu}(\mathrm{bnd}(w))\right) d\mathbb{P}_{w_1}(w) \\ &= \sum_{w_1}\mu(w_1)\int_\Omega \log\left(\frac{dw_1\nu}{d\nu}(\mathrm{bnd}(w))\right) d\mathbb{P}_{w_1}(w) \\ &= \sum_{w_1}\mu(w_1)\int_\Omega -\rho(w_1^{-1},\mathrm{bnd}(w))d\mathbb{P}_{w_1}(w) \end{aligned}$$

By the cocycle property, $\rho(w_1, w_1^{-1}\mathrm{bnd}(w)) = \rho(w_1, \mathrm{bnd}(\theta w))$ where θ is the shift on Ω. Since bnd is shift invariant, and $\omega_1 = g$

$$\begin{aligned} &= \sum_{w_1}\mu(w_1)\int_\Omega \rho(w_1,\mathrm{bnd}(w))d\mathbb{P}_{w_1}(w) \\ &= \sum_g \mu(g)\int_\Omega \rho(g,b)\,d\nu(b) \\ &= h_\mu(\Pi,\nu). \end{aligned}$$

■

Corollary 13 The Poisson boundary is trivial if and only if $h_{RW}(G,\mu) = 0$.

6.2.2 A Shannon-type theorem for the random walk entropy

Recall Kingman's subadditive ergodic theorem:

Theorem 14 Let (X,m) be a probability space and $T : X \to X$ a measure preserving transformation. Suppose that $f_n \in L^1(X,m)$ is a sub additive sequence, that is, $f_{n+m} \le f_n + f_m \circ T^n$ for all n,m. Then the limit $f = \lim \frac{1}{n} f_n \ge -\infty$ exists a.s. and f is T-invariant function.

If, furthermore, m is T-ergodic, then the limit f is the constant number which is $\inf_n \int \frac{1}{n} f_n(x)\,dm(x)$.

Let C_g^n be the cylinder of all ω such that $\omega_1\cdots\omega_n = g$.

Theorem 15 Let (G,μ) with $H(\mu) < \infty$. Then

$$\lim_{n\to\infty} -\frac{1}{n}\log\mu^n(w_n) = h_{RW}(G,\mu).$$

for a.e. ω, and as functions in L^1.

Proof. Here we want to consider the shift $\sigma : (G^{\mathbb{N}},\mu^{\mathbb{N}}) \to (G^{\mathbb{N}},\mu^{\mathbb{N}})$. Define $f_n(w) = -\log\mu^n(w_n)$. Write $w_n = g_1\cdots g_n$.

Note that $\mu^{n+m}(g_1\cdots g_n g_{n+1}\cdots g_{n+m}) \ge \mu^n(g_1\cdots g_n)\mu^m(g_{n+1}\cdots g_{n+m})$ and thus

$$\begin{aligned} f_{n+m}(w) &= -\log\mu^{n+m}(g_1\cdots g_n g_{n+1}\cdots g_{n+m}) \\ &\le -\log\mu^n(g_1\cdots g_n) - \log\mu^m(g_{n+1}\cdots g_{n+m}) \\ &= f_n(w) + f_m(\sigma^n w) \end{aligned}$$

where $\sigma : \Omega \to \Omega$ is the shift. Since the entropy is finite, $f_1 \in L^1(G,\mu)^{\mathbb{N}}$, and thus by the the subadditive ergodic theorem, the limit $\lim -\frac{1}{n}\log\mu^n(w_n)$ exists and it is equal to

$$\inf \frac{1}{n}\int -\log\mu^n(w_n)d\mu^{\mathbb{N}}(w) = \inf \frac{1}{n}H(\mu^n) = h_{RW}(G,\mu).$$

■

Corollary 14 Assume that there exist $\delta > 0$ and a sequence $A_n \subset G$ such that $\mu^n(A_n) > \delta$ and $\log|A_n| = o(n)$. Then $\Pi(G,\mu)$ is trivial.

Proof. Let $\phi_n(w) = -\frac{1}{n}\log\mu^n(w_n)$. So $\phi_n(w) \to h$ for almost every w.

For any n let $C_n \subset \Omega$ be the set of all walks that at time n visited at A_n. Let $\psi_n(w) = \phi_n(w) \cdot 1_{C_n}(w)$. So we get

$$\begin{aligned}\int \psi_n(w)d\mathbb{P}_\mu(w) &= -\frac{1}{n}\int_{C_n}\log\mu^n(w_n)d\mathbb{P}_\mu(w)\\ &= \frac{1}{n}H(\mu^n|Z_n \in A_n)\\ &\leq \frac{1}{n}\log|A_n| \to 0\end{aligned}$$

That is $\psi_n \to 0$ in L^1. Hence we can find n_k such that $\psi_{n_k}(w) \to 0$ almost surely. Let D be the full measure set of w for which $\psi_{n_k}(w) \to 0$ for all $w \in D$, and also $\phi_n(w) \to h$.

As a subsequence of numbers $\phi_{n_k}(w) \to h$ for all $w \in D$. Hence it is enough to find a positive measure set on which $\phi_{n_k}(w) \to 0$.

Now let C be the set of w such that $w \in C_n$ for infinitely many n's. That is, $C = \sup C_n$. Fatou's lemma says that $\mathbb{P}_\mu(C) \geq \limsup \mathbb{P}_\mu(C_n) \geq \delta$.

Now for any $w \in C \cap D$ we get that $\phi_{n_k}(w) = \psi_{n_k}(w) \to 0$.

■

Remark 13 We didn't use the fact that we are talking about iid random walk. In particular, if we have some other measure $\Lambda \in \text{Prob}(\Omega)$ with a Shannon theorem, that is $-\frac{1}{n}\log\Lambda(C^n_{w_n})$ convergence a.s. and in L^1 to some constant number, and we have a collection of sub exponentially family of sets A_n that is visited for Λ-almost every walk, then this constant is 0.

We can think on the collection A_n as guessing where the random walk is, at time n. The game here is to find the walker at time n, where we allow our guesses to be of sub-exponential size. We will soon use this idea to have a criterion when a certain boundary is actually the Poisson boundary.

6.2.3 A Shannon type theorem for Furstenberg entropy

Let (X,ν) be a G-space. Consider the one sided Bernoulli shift $(\Omega,\mathbb{P}) = (G,\mu)^{\mathbb{N}}$ with the shift $\theta : \Omega \to \Omega$ $\theta(\omega_1,\omega_2,\dots) = (\omega_2,\omega_3,\dots)$. And let $T : \Omega\times X \to \Omega\times X$ be $T(\omega,x) = (\theta\omega,\omega_1 x)$.

Claim 5 $\mathbb{P}\times\nu$ is T-invariant measure if and only if ν is μ-stationary. $\mathbb{P}\times\nu$ is ergodic if and only if ν is ergodic.

Proof. Since θ is ergodic, the only T-invariant sets are of the form $E\times\Omega$ where $E \subset X$. Thus $T^{-1}(\Omega\times E) = (\Omega\times(\mu * E))$, both parts are now follow ■

Theorem 16 Let (X,ν) be an ergodic (G,μ)-stationary space with finite entropy. Then If $h_\mu(X,\nu) = \lim_{n\to\infty} -\frac{1}{n}\log\frac{d(\omega_n\cdots\omega_1)^{-1}\nu}{d\nu}(x)$ for $\mathbb{P}\times\nu$ almost every (ω,x).

Proof. Recall the cocycle $\rho(g,x) = -\log \frac{dg^{-1}\nu}{d\nu}(x)$ and let $\bar{\rho}(\omega,x) = \rho(\omega_1,x)$.

The cocycle property, $\rho(\omega_2\omega_1,x) = \rho(\omega_2,\omega_1 x) + \rho(\omega_1,x)$, implies that

$$\begin{aligned}\bar{\rho}(\omega,x) + \bar{\rho}(T(\omega,x)) &= \rho(\omega_1,x) + \rho(\omega_2,\omega_1 x)\\ &= \rho(\omega_2\omega_1,x)\end{aligned}$$

and more generally, $\sum_{k=1}^{n} \bar{\rho}\left(T^k(\omega,x)\right) = \rho(\omega_n\cdots\omega_1,x)$.

Since we assume that the entropy is finite, $\bar{\rho}(\omega,x) = \rho(\omega_1,x) \in L^1(\mathbb{P}\times\nu)$. Now the ergodic theorem says that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \bar{\rho}\left(T^k(\omega,x)\right) = \int_{X\times\Omega} \bar{\rho}(\omega,x)\, d\mathbb{P}\times\nu = h_\mu(X,\nu).$$

Now,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} \bar{\rho}\left(T^k(x,\omega)\right) &= \frac{1}{n}\rho(\omega_n\cdots\omega_1,x)\\ &= -\frac{1}{n}\log\frac{d(\omega_n\cdots\omega_1)^{-1}\nu}{d\nu}(x).\end{aligned}$$

■

For boundaries we can have another limit. Note that here we take the positive direction of the random walk, evaluate the Radon Nikudym derivative in a specific point and no take the limit without the minus.

Lemma 28 Let (B,ν) be a boundary with finite entropy, then for almost every $w \in \mathbf{bnd}^{-1}(b)$

$$\frac{1}{n}\log\frac{dg_1\cdots g_n\nu}{d\nu}(b) \to h_\mu(B,\nu).$$

Proof. We want to show that $-\frac{1}{n}\rho((g_1\cdots g_n)^{-1},b)) \to h_\mu(B,\nu) = \sum_g \int \rho(g,b)d\nu(b)$. Lets calculate the expression:

$$\begin{aligned}\rho\left(g_n^{-1}\cdots g_1^{-1},b\right) &= \rho\left(g_n^{-1}\cdots g_2^{-1},g_1^{-1}b\right) + \rho\left(g_1^{-1},b\right)\\ &= \rho\left(g_n^{-1}\cdots g_3^{-1},g_2^{-1}g_1^{-1}b\right) + \rho\left(g_2^{-1},g_1^{-1}b\right) + \rho\left(g_1^{-1},b\right)\\ &= \rho\left(g_n^{-1}\cdots g_4^{-1},g_3^{-1}g_2^{-1}g_1^{-1}b\right) + \rho\left(g_3^{-1},g_2^{-1}g_1^{-1}b\right) + \rho\left(g_2^{-1},g_1^{-1}b\right) + \rho\left(g_1^{-1},b\right)\\ &= \sum_{k=1}^{n}\rho\left(g_k^{-1},g_{k-1}^{-1}\cdots g_1^{-1}b\right)\end{aligned}$$

Note that for $w \in \mathbf{bnd}^{-1}(b)$ we have that $g_{k-1}^{-1}\cdots g_1^{-1}\mathbf{bnd}(w) = \mathbf{bnd}(\sigma^{k-1}w) = \mathbf{bnd}(w)$, since $\mathbf{bnd}$ is σ-invariant function. Thus we get

$$-\frac{1}{n}\rho(g_n^{-1}\cdots g_1^{-1},\mathbf{bnd}(w)) = -\frac{1}{n}\sum_{k=1}^{n}\rho\left(g_k^{-1},\mathbf{bnd}(w)\right).$$

So we need to show that the right term convergence to the Furstenberg entropy.

Let $\varphi \in L^1\left(\Omega,\mu^{\mathbb{N}}\right)$ be the function $\varphi(w) = \rho(g_1,\mathbf{bnd}(\sigma(w)) = \rho\left(g_1,g_1^{-1}\mathbf{bnd}(w)\right) = -\rho\left(g_1^{-1},\mathbf{bnd}(w)\right)$.

Note that $\varphi(\sigma w) = -\rho\left(g_2^{-1}, \mathbf{bnd}(w)\right)$, since bnd is θ-invariant function. Now the ergodic average is

$$\frac{1}{n}\sum_{k=1}^{n}\varphi(\sigma^k w) = -\frac{1}{n}\sum_{k=1}^{n}\rho(g_k^{-1}, \mathbf{bnd}(w))$$

and by the ergodic theorem it convergence to $\int \varphi(w)d\mu^{\mathbb{N}}(w)$. We now verify that this is indeed the Furstenberg entropy of (B,ν):

$$\begin{aligned}
\int \varphi(w)d\mu^{\mathbb{N}}(w) &= \int \rho(g_1, \mathbf{bnd}(\sigma w))d\mu^{\mathbb{N}}(w) \\
&= \sum_g \mu(g)\int \rho(g, \mathbf{bnd}(\sigma w))d\mu^{\mathbb{N}}(w) \\
&= \sum \mu(g)\int_B \rho(g,b)\,d\nu(b) \\
&= h_\mu(B,\nu).
\end{aligned}$$

■

6.3 A criterion for the Poisson boundary

Let (B,ν) be a boundary. We want to get a tool that tells us whether (B,ν) is the Poisson boundary. Out tool is based on the entropy and so we will assume that $H(\mu) < \infty$.

Claim 6 If $(B_1,\nu_1) \to (B_2,\nu_2)$ is a measure preserving map between two boundaries, then they are equal.

Proof. There are several ways to see this fact. Recall that each boundary defines, via its Furstenberg transform a G-invariant closed sub algebra of $\mathscr{H}^\infty(G,\mu)$ and vice versa. Since the images of the Furstenbeg transforms coincide, they are isomorphic. ■

Corollary 15 A boundary (B,ν) is the Poisson boundary if and only if its entropy equals to the entropy of the Poisson boundary.

Recall that if we can guess the position of the random walk in a subexponential scale, then $h_{RW} = 0$. Now we apply this argument, for conditional entropy. That is, to show that B is the Poisson boundary we claim that it is enough to guess the position of the random walk in sub exponential scale, given the information of B. In other words, given the information that the random walk hits (through the boundary map) the point $b \in B$ we need to guess where the walker is, at time n.

Definition 24 A probability measure $\Lambda \in \mathscr{P}(\Omega)$ has **an asymptotic entropy** $h(\Lambda)$ if the limit

$$-\frac{1}{n}\log\Lambda\left(C^n_{g_1\cdots g_n}\right) \to h(\Lambda)$$

exist for a.e. $\omega \in \Omega$ and in $L^1(\Lambda)$.

The Shannon-type theorem that we proved showed that the Markov measure of the μ-random walk has an asymptotic entropy (the Markov measure is the measure on Ω when we think of it as the space of walks and not increments).

Let (B,ν) be a μ-boundary. For any $b \in B$, we can take the Markov chain on G which is conditioned to hit b. We denote this measure on Ω by $\mathbb{P}^b$.

Recall that given $E \subset B$ with $\nu(E) > 0$ and $C_{e,g_1,\ldots,g_n} \subset \Omega$ be a cylinder. Then

$$\mathbb{P}\left(C_{e,g_1,\ldots,g_n}\cap \mathrm{bnd}^{-1}E\right)=\mathbb{P}\left(C_{e,g_1,\ldots,g_n}\right)\mathbb{P}_g\left(\mathrm{bnd}^{-1}E\right)=\mathbb{P}\left(C_{e,g_1,\ldots,g_n}\right)g\nu\left(E\right).$$

where $\mathbb{P}_g$ is the Markov chain that starts from g instead of e. Now

$$\mathbb{P}\left(C_{e,g_1,\ldots,g_n}|\mathrm{bnd}^{-1}E\right)=\mathbb{P}\left(C_{e,g_1,\ldots,g_n}\right)\frac{g\nu\left(E\right)}{\nu\left(E\right)}.$$

Since it holds for any E, we can write the density of this measure for a single $b\in B$ (for a.e. b):

$$\mathbb{P}_\mu^b(C^n_{g_1,\ldots,g_n})=\mathbb{P}_\mu(C^n_{g_1,\ldots,g_n})\frac{dg_1,\ldots,g_n\nu}{d\nu}(b).$$

Lemma 29 Let (B,ν) be a (G,μ)-boundary. Then for a.e. $b\in B$, the measure $\mathbb{P}^b$ has an asymptotic entropy and it is equal to

$$h\left(\mathbb{P}^b\right)=h_{RW}\left(G,\mu\right)-h_\mu\left(B,\nu\right).$$

Proof. Apply log on $\mathbb{P}_\mu^b(C^n_{g_1,\ldots,g_n})=\mathbb{P}_\mu(C^n_{g_1,\ldots,g_n})\frac{dg_1\cdots g_n\nu}{d\nu}(b)$ and let $w\in\mathbf{bnd}^{-1}(b)$. Then

$$\begin{aligned}-\frac{1}{n}\log\mathbb{P}_\mu^b(C^n_{g_1,\ldots,g_n}) &= -\frac{1}{n}\log\mathbb{P}_\mu(C^n_{g_1\cdots g_n})-\frac{1}{n}\log\frac{dg_1\cdots g_n\nu}{d\nu}(b)\\ &\to h_{RW}\left(G,\mu\right)-h_\mu\left(B,\nu\right)\end{aligned}$$

■

Corollary 16 A boundary (B,ν) is the Poisson boundary if and only if $h(\mathbb{P}_\mu^b)=0$ for almost every $b\in B$.

In particular, if (B,ν) is a μ-boundary, and there exist some $\delta>0$ and a sequence of maps $\pi_n:B\to 2^G$ such that $\mathbb{P}_\mu^b(C^n_{\pi_n(b)})>\delta$ and $\log\left|\pi_n\left(b\right)\right|=o\left(n\right)$, then (B,ν) is the Poisson boundary.

6.4 Furstenberg's entropy realization problem

Problem 6.1 Given (G,μ) find all values in $[0,h_{RW}(G,\mu)]$ obtained as the Furstenberg entropy of ergodic μ-stationary actions.

6.4.1 Entropy gap

Consider the classical settings of ergodic theory, that is $G\curvearrowright(X,m)$ is a probability measure preserving action.

Consider $L^2\left(X,m\right)$. We have a G-action on $L^2\left(X,m\right)$, by $gf\left(x\right)=f\left(g^{-1}x\right)$. Note that this is a unitary representation of G:

$$\|gf\|_2^2=\int\left|f\left(g^{-1}x\right)\right|^2dm\left(x\right)=\int\left|f\left(x\right)\right|^2dg^{-1}m\left(x\right)=\|f\|_2^2.$$

It is a very useful tool. For example, one can define ergodicity as the only G-invariant subspace is the one dimensional space of constant functions. Similarly weakly mixing can also be formulated via this Koompan representation.

For non measure preserving measures the action is no longer unitary. However, we can fix it. The following works for quasi invariant measures, but we can think only on stationary actions $G \curvearrowright (X,\nu)$.

Let $\pi : G \to \mathscr{U}(L^2(X,\nu))$ be the unitary representation defined by:

$$\pi(g) f(x) = f\left(g^{-1}x\right)\left(\frac{dg\nu}{d\nu}(x)\right)^{\frac{1}{2}}.$$

And then

$$\begin{aligned}\|\pi(g) f\|_2^2 &= \int \left|f\left(g^{-1}x\right)\right|^2 \frac{dg\nu}{d\nu}(x)\, d\nu(x)\\ &= \int \left|f\left(g^{-1}x\right)\right|^2 dg\nu(x)\\ &= \int |f(x)|^2 d\nu(x)\\ &= \|f\|_2^2\end{aligned}$$

However, the situation becomes quite different than the one in the measure preserving setup. For example, even the one dimensional subspace of constant functions is no longer invariant.

Lemma 30 Let $G \curvearrowright (X, nu)$ be an ergodic action. If there exists a non-invariant vector in $L^2(X,\nu)$ then ν is equivalent to a G-invariant probability measure.

Proof. Let f be an invariant function. So $f(x) = f(g^{-1}x)\frac{dg\nu}{d\nu}(x)^{\frac{1}{2}}$ for a.e. x. Let $S = \{x | f(x) \neq 0\}$ the support of f. So S is non invariant G-invariant set hence, by ergodicity $\nu(S) = 1$. Now define $d\eta(x) = f(x)^2 d\nu(x)$. Since S has full measure, the new measure, η, is equivalent to ν.

Now,

$$d\eta(g^{-1}x) = f(g^{-1}x)^2 d\nu(g^{-1}x) = f(g^{-1}x)^2\frac{dg\nu}{d\nu}(x)d\nu(x) = f(x)^2 d\nu(x) = d\eta(x)$$

that is, η is a G-invariant measure. It has finite mass since $\eta(X) = \int 1 d\eta(x) = \int f(x)^2 d\nu(x) = \|f\|^2$. By normalizing f we can make η a G-invariant probability measure. ■

Corollary 17 Let (X,ν) be an ergodic non-invariant stationary action. Then $L^2(X,\nu)$ admits no non-zero invariant vectors.

We now turn to relate this representation with the Furstenberg entropy. Note that

$$\langle \pi(g) 1, 1\rangle = \int \left(\frac{dg\nu}{d\nu}\right)^{\frac{1}{2}}(x)d\nu(x)$$

and so

$$\begin{aligned}\varphi(g) &= -2\int_X \log\left(\frac{dg^{-1}\nu}{d\nu}(x)\right)^{\frac{1}{2}}(x)\, d\nu(x)\\ &\geq -2\log\int_X \left(\frac{dg^{-1}\nu}{d\nu}(x)\right)^{\frac{1}{2}}(x)\, d\nu(x)\\ &= -2\log\left\langle \pi\left(g^{-1}\right)1,1\right\rangle\\ &= -2\log\langle 1,\pi(g)1\rangle\end{aligned}$$

Let $\pi(\mu): L^2(X,\nu) \to L^2(X,\nu)$ be the operator

$$\pi(\mu)(f) = \sum \mu(g)\pi(g) f.$$

We get

$$\begin{aligned} h_\mu(X,\nu) &= \sum_g \mu(g)\varphi(g) \\ &\geq \sum \mu(g) - 2\log\langle 1, \pi(g)1\rangle \\ &= -2\log \sum \mu(g)\langle 1, \pi(g)1\rangle \\ &\geq -2\log\|\pi(\mu\| \end{aligned}$$

Now define

$$\|\mu\|_T = \sup\left\{ \|\pi(\mu)\| : \pi \text{ unitary representation with no invariant vectors } \right\}$$

Definition 25 If G has property (T) then $\|\mu\|_T < 1$ for every fully supported measure measure μ.

The original definition says that G has property (T) if any representation that admits almost invariant vector, admits an invariant vector.

Typical (discrete) examples are the groups $SL_n(\mathbb{Z})$ for $n \geq 3$. If G is amenable group that has also property (T) then G is finite (compact).

A quotient of property (T) groups has property (T) and hence the product of property (T) group with non property (T) doesn't have property (T).

Corollary 18 If G has property (T) then for any generating random walk μ, there exists an entropy gap. That is, there exists $C_\mu > 0$ such that any non invariant ergodic stationary space has entropy greater than C_μ.

Note that there exists groups without property (T) with an entropy gap - for example $\mathbb{Z} \times G$ where G has (T).

However, it is interesting to understand the class of groups with an entropy gap. For example - is it true that a simple group without (T) cannot have gap?

The only know group that are known to have no gap, for any measure with finite first moment are lamplighters over recurrent base group, free groups and virtually free groups, as $SL_2(\mathbb{Z})$.

6.5 Poisson bundles

$Cos_G = \{Hg\}$. Note that $Hg = gg^{-1}Hg = gH^{g-1}$ The advantage is that we get a left action

6.6 Identifications of Poisson boundaries

The lamplighter group.

6.6.1 Group compactification

For some classes of groups there are "natural" boundaries, which comes from the geometry of the group. Some of these spaces were proved to be the Poisson boundary for natural measures. Let us mentioned some examples.

Let M be a metric space. By a **compactification** of M we mean a compact Hausdorff space $M \subset \bar{M}$ which is second countable (there exists a countable basis for the topology) such that M open and dense in $\bar{M}$. The boundary of M is $\partial M = \bar{M} \backslash M$.

Now we consider M to be a group. But then, we require the compactification to be compatible with the group action:

Let G be a countable group. We say that $\bar{G} = G \cup \partial G$ is a **compatible compactification** if the left action of G on itself extends to an action of G on $\bar{G}$ by homeomorphisms.

Definition 26 A compactification is said to be non-elementary if there is no finite set in ∂G which is fixed by G.

A (non recurrent) random walk will convergence to an element in ∂G. Thus we get a map $\Pi : (\Omega, \mathbb{P}) \to \partial G$. It is clear that $\nu = \Pi_* \mathbb{P}$ is stationary and moreover, since Π is shift invariant, so $(\partial G, \nu)$ is a boundary.

6.6.2 Ends of a group

For a compact subset K is a locally compact space M we denote by $\mathscr{E}_K(M)$ the set of the connected components of $M \backslash K$. If $K_1 \subset K_2$ then there is a natural map $\mathscr{E}_{K_2}(M) \to \mathscr{E}_{K_1}(M)$. The projective limit is the space of ends as K exhaust the whole space M is denoted by $\mathscr{E}(M)$. The compactification $\bar{M} = M \cup \mathscr{E}(M)$ is called the **ends compactification**.

If G is a finitely generated group then we can play this game on the Cayley graph of G.

Example: $\mathbb{Z}$ has 2 ends. $\mathbb{Z}^2$ has one. For the free group, $\mathscr{E}(F_2)$ is the set of infinite reduced words.

6.6.3 Gromov boundary

Definition 27 Let M be a geodesic metric space. A geodesic triangle (or, three points) are **δ-thin** if the distance of any point on any of the edges to the two other edges is at most δ.

M is called **δ-hyperbolic** is every triangle is δ-thin.

M is called **hyperbolic** if it is δ hyperbolic for some $\delta \geq 0$.

Pick some origin $o \in M$. A geodesic ray from o is an isometry $\gamma : [0, \infty) \to M$ such that, $\gamma(0) = o$ and the path $\gamma([0,t])$ is a geodesic connected o and $\gamma(t)$, for any t.

We say that γ_1 and γ_2 are equivalent if they stay within a bounded distance from each other. That is, there exist K such that $d(\gamma_1(t), \gamma_2(t)) \leq K$ for all t.

The Gromov boundary of M is the set of equivalence classes of geodesic rays.

Now play this game on a group. The Gromoc boundary is a compatible compactification.

Example: The free group.

Theorem Let G be a hyperbolic group or a group with infinitely many ends, and let μ be a generating random walk. Then if μ has finite first moment then $(\partial G, \Pi_* \mathbb{P})$ is the Poisson boundary.

6.6.4 Lamplighter groups

6.6.5 The Affine group

Consider the affine group $ax + b$. That is, each element g is of the form (a_g, b_g) and the multiplication is composition of maps.

That is, gh is the map

$$a_g(a_h x + b_h) + b_g = a_g a_h x + (a_g b_h + b_g).$$

A nice way to represent this group is by $ax + b \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and the action is on the vectors of the form $\begin{pmatrix} x \\ 1 \end{pmatrix}$.

Now lets consider $Aff\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$, that is the set of elements $(x,f)=\begin{pmatrix} 2^x & f \\ 0 & 1 \end{pmatrix}$ where $f=\frac{m}{2^n}$. We can see that

$$(x_1,f_1)(x_2,f_2)=(x_1+x_2,f_1+2^{x_1}f_2)$$

and in general

$$(x_1,f_1)\cdots(x_n,f_n)=(x_1+\cdots+x_n,f_1+2^{y_1}f_2+\cdots+2^{y_{n-1}}f_n).$$

Consider the homomorphism $Aff\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)\to\mathbb{Z}$, $(x,f)\mapsto x$. Given μ on $Aff\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$ let $\mu_{\mathbb{Z}}\in\mathscr{P}(\mathbb{Z})$ be the projected random walk. Write $\alpha=\mathbb{E}\mu_{\mathbb{Z}}$ for the drift.

This group is finitely generated by the two elements

$$b=\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}=(1,0),\quad a=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}=(0,1)$$

with the relation $a^2b=ba$. Which falls into the Baumslag-Solitar groups format: $BS(1,2)$. In general, $BS(n,m)=\langle a,b|a^nb=ba^m\rangle$.

Let $\pi:LL(\mathbb{Z},\mathbb{Z})\to Aff\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$ be the map

$$\pi(x,f)=\left(x,\sum_k 2^k f(k)\right).$$

We can see that

$$K=\ker\pi=\left\{(x,f)\,|x=0,\sum_k 2^k f(k)=0\right\}.$$

and so $LL(\mathbb{Z},\mathbb{Z})/K=Aff\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$. In particular, for a measure $\mu\in\mathscr{P}(L)$ and projected measure $\bar{\mu}\in\mathscr{P}(A)$ we have that $\Pi(A,\bar{\mu})=K\backslash\Pi(L,\mu)$.

In particular, if $\Pi(L,\mu)$ is trivial, so do $\Pi(A,\bar{\mu})$.

Let μ be the SRW. Then clearly, $\Pi(L,\mu)$ is trivial and so do $\Pi(A,\bar{\mu})$.

Theorem 17 Let $\mu\in\mathscr{P}\left(Aff\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)\right)$ be generating with finite first moment. Let $\mu_{\mathbb{Z}}$ be the projected random walk on $\mathbb{Z}$ via the map $(x,f)\mapsto x$ and let $\alpha=\sum z\mu_{\mathbb{Z}}(z)$. Then

1. If $\alpha<0$ then for almost every path (y_n,φ_n) the limit exists $\lim\varphi_n=\varphi_\infty\in\mathbb{R}$. And $\mathbb{R}$ with the distribution of φ_∞ is the Poisson boundary.
2. If $\alpha=0$ then the Poisson boundary is trivial.
3. If $\alpha>0$ then the limit $\lim\varphi_n=\varphi_\infty\in\mathbb{Q}_2$ (numbers of the form $\frac{a}{2^b}$) in the 2-adic topology.

$|x|_2=2^{-n}$ where $x=2^n\cdot q$ where q is odd, the metric induced by this norm.