

$C_0(X)$ -ALGEBRAS, STABILITY AND STRONGLY SELF-ABSORBING C^* -ALGEBRAS

ILAN HIRSHBERG, MIKAEL RØRDAM, AND WILHELM WINTER

ABSTRACT. We study permanence properties of the classes of stable and so-called \mathcal{D} -stable C^* -algebras, respectively. More precisely, we show that a $C_0(X)$ -algebra A is stable if all its fibres are, provided that the underlying compact metrizable space X has finite covering dimension or that the Cuntz semigroup of A is almost unperforated (a condition which is automatically satisfied for C^* -algebras absorbing the Jiang–Su algebra \mathcal{Z} tensorially). Furthermore, we prove that if \mathcal{D} is a K_1 -injective strongly self-absorbing C^* -algebra, then A absorbs \mathcal{D} tensorially if and only if all its fibres do, again provided that X is finite-dimensional. This latter statement generalizes results of Blanchard and Kirchberg. We also show that the condition on the dimension of X cannot be dropped. Along the way, we obtain a useful characterization of when a C^* -algebra with weakly unperforated Cuntz semigroup is stable, which allows us to show that stability passes to extensions of \mathcal{Z} -absorbing C^* -algebras.

0. INTRODUCTION

This paper addresses stability and \mathcal{D} -stability of $C_0(X)$ -algebras, and related matters. Recall ([34]) that a separable, unital, infinite dimensional C^* -algebra \mathcal{D} is said to be *strongly self-absorbing* if the embedding $\mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D}$ given by $d \mapsto d \otimes 1$ is approximately unitarily equivalent to an isomorphism (regardless of which tensor product is used in the definition, such a \mathcal{D} is automatically nuclear, and therefore there is no ambiguity in the definition). The list of known examples of strongly self-absorbing C^* -algebras is very short: it consists of UHF algebras of ‘infinite type’ (i.e., ones where all the primes which appear in the supernatural number do so with infinite multiplicity), the Jiang–Su algebra \mathcal{Z} ([18]), the Cuntz algebras \mathcal{O}_2 and \mathcal{O}_∞ , and tensor products of \mathcal{O}_∞ by UHF algebras of infinite type.

We note that the algebra of compact operators on a separable Hilbert space, \mathcal{K} , is not strongly self-absorbing (it is not unital), but it does have some similar properties. In particular, tensoring by a strongly self-absorbing C^* -algebra \mathcal{D} can be seen as an analogue of stabilization. We shall thus refer to a C^* -algebra A as being *\mathcal{D} -stable*, or *\mathcal{D} -absorbing*, if $A \cong A \otimes \mathcal{D}$. \mathcal{D} -stability, for the known examples of strongly self-absorbing C^* -algebras, has been studied for quite some time; the concept is of particular interest for Elliott’s program to classify nuclear C^* -algebras

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by their K -theory data (see [29] for an introduction). The case of UHF algebras was studied by the second-named author in [26, 27]. Absorption of \mathcal{O}_2 and of \mathcal{O}_∞ plays an important role in the classification results of Kirchberg and Phillips ([23, 22, 25]), and has been the focus of further study (see, e.g., [24, 21]). Absorption of the Jiang–Su algebra seems to be particularly important, since it is automatic for the known classifiable algebras, whereas the known counterexamples to the Elliott conjecture fail to absorb \mathcal{Z} . Algebras which absorb the Jiang–Su algebra tend to be well-behaved in many respects, and thus, \mathcal{Z} -stability can be seen as a regularity property, which needs to be better understood (see [11, 33, 35, 36, 37, 38]). We shall comment further on this below.

Stability and \mathcal{D} -stability have different permanence properties. It was shown in [15] that stability for σ -unital C^* -algebras passes to inductive limits and crossed products by discrete groups. Stability clearly does not in general pass to hereditary subalgebras (look at corners of the compact operators), and it was even shown in [28] that there are examples of non-stable C^* -algebras A where $M_n(A)$ is stable for some n . Stability passes to quotients and ideals, however an extension of one stable C^* -algebra by another need not be stable – see [30]. We refer the reader to [32] for a more recent survey and further details. As for \mathcal{D} -stability, it was shown in [34], under the mild extra condition of K_1 -injectivity (i.e., the canonical map from $\mathcal{U}(\mathcal{D})/\mathcal{U}_0(\mathcal{D})$ to $K_1(\mathcal{D})$ is injective – a property which holds for all known examples of strongly self-absorbing C^* -algebras), that \mathcal{D} -stability passes to hereditary subalgebras, quotients, inductive limits and extensions of separable C^* -algebras. Those results have been shown earlier by Kirchberg for the cases of $\mathcal{D} = \mathcal{O}_2$ and $\mathcal{D} = \mathcal{O}_\infty$, in [20]. \mathcal{D} -stability does not pass in general to crossed products; however, it does pass to crossed products by \mathbb{Z} , \mathbb{R} or by compact groups provided the group action has a Rokhlin property ([14]).

The main questions addressed in this paper concern the behavior of bundles of stable or \mathcal{D} -stable algebras. Example 3.11 (from [28]) exhibits a non-stable C^* -algebra, which arises as a continuous field of C^* -algebras, with fibres all isomorphic to the compacts. The base space in that example is infinite dimensional. We shall show, in Section 3, that if X is a locally compact Hausdorff space of finite covering dimension, and if A is a $\mathcal{C}_0(X)$ -algebra such that all its fibres are stable, then so is A . Thus, infinite dimensionality is crucial in Example 3.11. We also obtain another regularity property of \mathcal{Z} -stability: If A is a $\mathcal{C}_0(X)$ -algebra (this time with no restriction on the dimension of X), such that A is \mathcal{Z} -stable, and all the fibres are stable, then A is stable as well. To show the latter, we use the fact (shown in [33]) that the Cuntz semigroup of a \mathcal{Z} -stable C^* -algebra is almost unperforated, and along the way, obtain a characterization of stability of such C^* -algebras. This characterization also allows us to show that stability is preserved under forming extensions, assuming the algebra in the middle has almost unperforated Cuntz semigroup. We also use the above results to show that if A is \mathcal{Z} -stable, and $M_n(A)$ is stable for some n , then so is A .

In Section 4, we prove our main result that if \mathcal{D} is a K_1 -injective strongly self-absorbing C^* -algebra, A is a separable $\mathcal{C}_0(X)$ algebra, X is of finite covering dimension and all the fibres of A are \mathcal{D} -stable, then A is also \mathcal{D} -stable. This generalizes earlier results of Blanchard and Kirchberg ([6]), who prove a similar statement under the additional requirements that $\mathcal{D} = \mathcal{O}_\infty$, $X = \text{Prim}(A)$, and A is nuclear

and stable (with different methods; see also [5] for related results). The theorem fails in the case of an infinite-dimensional base space X , as is shown in Examples 4.7 and 4.8; cf. also [9]. In the first example, we construct a unital $\mathcal{C}(X)$ -algebra (where $X = \prod_{\mathbb{N}} S^2$) whose fibres are isomorphic to the UHF-algebra of type 2^∞ (the CAR-algebra) but which itself does not absorb the CAR-algebra tensorially (in fact, it does not admit a unital embedding of M_2). In the second, we give a similar example where the fibres again are isomorphic to the CAR-algebra, and hence are \mathcal{Z} -stable, but where the algebra itself is not \mathcal{Z} -stable. On the other hand we show that if X is not of finite covering dimension, but if A is *locally* \mathcal{D} -stable, then A is also \mathcal{D} -stable.

In some special cases, much stronger results than ours hold: in a very recent paper by Dădărlat, [8], it is shown (as a corollary to a more general result) that if A is a $\mathcal{C}(X)$ -algebra over a finite-dimensional compact space X such that each fibre is isomorphic to \mathcal{D} – where \mathcal{D} is either \mathcal{O}_2 or \mathcal{O}_∞ – then A is isomorphic to $\mathcal{C}(X) \otimes \mathcal{D}$ (the latter clearly implies that A is \mathcal{D} -stable). In [10], Dădărlat and the third named author derive the respective result for an arbitrary K_1 -injective strongly self-absorbing C^* -algebra.

The paper is organized as follows: in Sections 1 and 2 we recall some facts about $\mathcal{C}_0(X)$ -algebras and about C^* -algebras with almost unperforated Cuntz semigroup. In Section 3 we characterize when such algebras are stable and prove our results about stability of $\mathcal{C}_0(X)$ -algebras and about extensions. Section 4 is entirely devoted to \mathcal{D} -stability of $\mathcal{C}_0(X)$ -algebras.

1. $\mathcal{C}_0(X)$ -ALGEBRAS

We recall some facts and notation about $\mathcal{C}_0(X)$ -algebras. This is a concept introduced by Kasparov to generalize continuous bundles (or fields) of C^* -algebras over locally compact Hausdorff spaces, cf. [19].

1.1 DEFINITION: *Let A be a C^* -algebra and X a locally compact σ -compact Hausdorff space. A is a $\mathcal{C}_0(X)$ -algebra, if there is a $*$ -homomorphism $\mu: \mathcal{C}_0(X) \rightarrow \mathcal{Z}(\mathcal{M}(A))$ from $\mathcal{C}_0(X)$ to the center of the multiplier algebra of A such that, for some (or any) approximate unit $(h_\nu)_{\nu \in \mathbb{N}}$ of $\mathcal{C}_0(X)$, $\|\mu(h_\nu) \cdot a - a\| \rightarrow 0$ for each $a \in A$.*

The map μ is called the *structure map*. We will usually not write it explicitly. Note that if X is compact, then μ has to be unital. In the compact case, we may write “ $\mathcal{C}(X)$ -algebra” instead of “ $\mathcal{C}_0(X)$ -algebra”.

1.2 If A is as above and $Y \subset X$ is a closed subset, then

$$J_Y := \mathcal{C}_0(X \setminus Y) \cdot A$$

is a (closed) two-sided ideal of A ; we denote the quotient map by π_Y and set

$$A_Y := A/J_Y.$$

If $a \in A$, we sometimes write a_Y for $\pi_Y(a)$. If Y consists of just one point x , we will slightly abuse notation and write A_x, J_x, π_x and a_x in place of $A_{\{x\}}, J_{\{x\}}, \pi_{\{x\}}$ and $a_{\{x\}}$, respectively. We say that A_x is the *fibre of A at x* .

1.3 For any $a \in A$ and for any $f \in \mathcal{C}_0(X)$ we have $\pi_x(f \cdot a) = f(x)\pi_x(a)$ (because $(fh_\nu - f(x)h_\nu)a$ belongs to J_x for all h_ν in an approximate unit for $\mathcal{C}_0(X)$).

Moreover,

$$\|a_x\| = \inf\{\|(1 - f(x))a + f \cdot a\| \mid f \in \mathcal{C}_0(X)\}$$

for every $x \in X$. The function $x \mapsto \|a_x\|$ from X to \mathbb{R} , being the infimum of a family of continuous functions from X to \mathbb{R} , is upper semicontinuous. That is, for any $x_0 \in X$ and $\varepsilon > 0$, there is a neighborhood $V \subset X$ of x_0 such that $\|a_x\| < \|a_{x_0}\| + \varepsilon$ for any $x \in V$. Equivalently, the set $\{x \in X \mid \|a_x\| < \varepsilon\}$ is open for every $a \in A$ and every $\varepsilon > 0$.

1.4 The family $\{\pi_x \mid x \in X\}$ is faithful, i.e., if $a \in A$ is such that $\|a_x\| = 0$ for all $x \in X$, then $a = 0$. Indeed, if $a \in A$ is a positive element, the C^* -algebra $B := C^*(\mu(\mathcal{C}_0(X)), a) \subset \mathcal{M}(A)$ is commutative. Therefore, there is a character χ on B such that $\chi(a) = \|a\|$. On the other hand, since A is a $\mathcal{C}_0(X)$ -algebra, there is $f \in \mathcal{C}_0(X)$ such that $\|f \cdot a - a\|$ is small, which shows that $\chi|_{\mathcal{C}_0(X)}$ is nonzero, hence a character on $\mathcal{C}_0(X)$. But then χ annihilates $\mathcal{C}_0(X \setminus \{x\})$ for some $x \in X$, so it drops to a character $\bar{\chi}$ on $\pi_x(C^*(a))$ satisfying $\chi(a) = \bar{\chi} \circ \pi_x(a)$.

It follows that the map $A \rightarrow \prod_{x \in X} A_x$ is injective, whence

$$\|a\| = \sup\{\|a_x\| \mid x \in X\}$$

for every $a \in A$.

1.5 If A is a $\mathcal{C}_0(X)$ -algebra and $Y \subset X$ is a closed subset, then the quotient A_Y may be regarded as a $\mathcal{C}_0(Y)$ -algebra by $g \cdot a_Y := (\bar{g} \cdot a)_Y$ for $a \in A$, $g \in \mathcal{C}_0(Y)$ and some lift $\bar{g} \in \mathcal{C}_0(X)$ of g .

If $f \in \mathcal{C}_0(X)$ is a function with support in Y , then the closure of $f|_Y \cdot A_Y$ is a $\mathcal{C}_0(Y)$ -algebra which embeds isometrically into A via

$$f|_Y \cdot a_Y \mapsto f \cdot a$$

for $a \in A$.

1.6 We are indebted to E. Blanchard for pointing out the following result about tensor products of $\mathcal{C}_0(X)$ -algebras to us (see [4, Proposition 3.1] – Blanchard states the result only for compact X , but the version below follows immediately by passing to the one-point compactification). As usual, we denote the minimal tensor product of A and B by $A \otimes B$.

PROPOSITION: *Let X and Y be locally compact spaces, A a $\mathcal{C}_0(X)$ -algebra and B a $\mathcal{C}_0(Y)$ -algebra. Suppose that either A or B are exact. Then, $A \otimes B$ is a $\mathcal{C}_0(X \times Y)$ -algebra with fibres*

$$(A \otimes B)_{(x,y)} \cong A_x \otimes B_y$$

for $x \in X$, $y \in Y$ and structure map μ given by

$$\mu = \mu_A \otimes \mu_B: \mathcal{C}_0(X) \otimes \mathcal{C}_0(Y) \rightarrow \mathcal{M}(A \otimes B).$$

REMARK: Using [1, Proposition IV.3.4.23], one can replace the assumptions on A (or B) by asking each fibre A_x (or B_x) of A (or of B) to be nuclear.

1.7 PROPOSITION: *Let $(B_n, \varphi_{n,n+1})$ and $(\mathcal{C}(Y_n), \gamma_{n,n+1})$ be unital inductive systems of C^* -algebras with limits A and $\mathcal{C}(X)$, respectively. Suppose each B_n is a $\mathcal{C}(Y_n)$ -algebra with (unital) structure maps $\mu_n: \mathcal{C}(Y_n) \rightarrow \mathcal{Z}(B_n)$ satisfying*

$$(1) \quad \varphi_{n,n+1} \circ \mu_n = \mu_{n+1} \circ \gamma_{n,n+1}$$

for all n . Then, A is a $\mathcal{C}(X)$ -algebra with (unital) structure map $\mu: \mathcal{C}(X) \rightarrow \mathcal{Z}(A)$ satisfying

$$\varphi_{n,\infty} \circ \mu_n = \mu \circ \gamma_{n,\infty}$$

for all n . If $x \in X = \lim_{\leftarrow} Y_n$ is a point corresponding to a sequence $\{y_n\}_{n \in \mathbb{N}} \in \prod Y_n$, then

$$A_x \cong \varinjlim (B_n)_{y_n}.$$

PROOF: The compatibility condition (1) guarantees existence of the unital *-homomorphism $\mu: \mathcal{C}(X) \rightarrow A$. Since each μ_n maps to $\mathcal{Z}(B_n)$, it is clear that μ also maps to the center of A , whence A is a $\mathcal{C}(X)$ -algebra.

Now if $x \in X$, then $\text{ev}_x \circ \gamma_{n,\infty}$ is a character on $\mathcal{C}(Y_n)$, hence corresponds to a point evaluation ev_{y_n} for some $y_n \in Y_n$. It is clear that

$$\gamma_{n,n+1}(\mathcal{C}_0(Y_n \setminus \{y_n\})) \subset \mathcal{C}_0(Y_{n+1} \setminus \{y_{n+1}\})$$

and that

$$\mathcal{C}_0(X \setminus \{x\}) = \varinjlim \mathcal{C}_0(Y_n \setminus \{y_{n+1}\}).$$

From (1) it follows that

$$\varphi_{n,n+1}(\mu_n(\mathcal{C}_0(Y \setminus \{y_n\}))) \cdot B_n \subset \mu_{n+1}(\mathcal{C}_0(Y_{n+1} \setminus \{y_{n+1}\})) \cdot B_{n+1},$$

so each $\varphi_{n,n+1}$ induces a *-homomorphism

$$\bar{\varphi}_{n,n+1}: (B_n)_{y_n} \rightarrow (B_{n+1})_{y_{n+1}}.$$

The maps $\pi_x \circ \varphi_{n,\infty}: B_n \rightarrow A_x$ clearly induce a *-homomorphism

$$\bar{\pi}_x: \varinjlim (B_n)_{y_n} \rightarrow A_x.$$

Conversely, the maps $\pi_{y_n}: B_n \rightarrow (B_n)_{y_n}$ induce a *-homomorphism

$$\psi: A = \varinjlim B_n \rightarrow \varinjlim (B_n)_{y_n}.$$

Since $\mu(\mathcal{C}_0(X \setminus \{x\})) \cdot A = \lim_{\rightarrow} \mu_n(\mathcal{C}_0(Y_n \setminus \{y_n\})) \cdot B_n$, we have

$$\begin{aligned} \psi(\ker \pi_x) &= \psi(\mu(\mathcal{C}_0(X \setminus \{x\})) \cdot A) \\ &= \psi(\varinjlim \mu_n(\mathcal{C}_0(Y_n \setminus \{y_n\})) \cdot B_n) \\ &= \psi(\varinjlim \ker \pi_{y_n}) \\ &= 0. \end{aligned}$$

Therefore, ψ induces a *-homomorphism

$$\psi_x: A_x \rightarrow \varinjlim (B_n)_{y_n};$$

it is routine to check that $\bar{\pi}_x$ and ψ_x are mutual inverses. \square

REMARK: With a little extra effort one can prove a nonunital version of the preceding result, replacing unitality of the connecting maps by certain nondegeneracy conditions.

1.8 In Section 4 we shall have use for the following combination of Propositions 1.6 and 1.7.

LEMMA: *Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of exact separable unital C^* -algebras such that each A_n is a $\mathcal{C}(X_n)$ -algebra. Then $A = \bigotimes_{n=1}^{\infty} A_n$ is a $\mathcal{C}(X)$ -algebra, where $X = \prod_{n=1}^{\infty} X_n$, with fibres*

$$A_x = \bigotimes_{n=1}^{\infty} (A_n)_{x_n},$$

for each $x = (x_1, x_2, \dots)$ in X .

PROOF: Set $B_n := \bigotimes_{k=1}^n A_k$ and define $\varphi_{n,n+1}: B_n \rightarrow B_{n+1}$ by $\varphi_{n,n+1} := \text{id}_{B_n} \otimes \mathbf{1}_{A_{n+1}}$. Similarly, define $Y_n := \prod_{k=1}^n X_k$ and, identifying $\mathcal{C}(Y_n)$ with $\bigotimes_{k=1}^n \mathcal{C}(X_k)$, define $\gamma_{n,n+1}: \mathcal{C}(Y_n) \rightarrow \mathcal{C}(Y_{n+1})$ by $\gamma_{n,n+1} := \text{id}_{\mathcal{C}(Y_n)} \otimes \mathbf{1}_{X_{n+1}}$.

Repeated applications of Proposition 1.6 show that each B_n is a $\mathcal{C}(Y_n)$ -algebra with structure map $\mu_n: \mathcal{C}(Y_n) \rightarrow \mathcal{Z}(B_n)$ given by $\bigotimes_{k=1}^n \nu_k$, where the $\nu_k: \mathcal{C}(X_k) \rightarrow \mathcal{Z}(A_k)$ are the structure maps of the $\mathcal{C}(X_k)$ -algebras A_k .

The μ_k clearly satisfy

$$\varphi_{n,n+1} \circ \mu_n = \mu_{n+1} \circ \gamma_{n,n+1}.$$

Therefore, Proposition 1.7 applies and A is a $\mathcal{C}(X)$ -algebra with structure map μ compatible with the inductive limit structure. Here, we identify $X = \prod X_n$ with colim $Y_n \subset \prod Y_n$. Now if $x = (x_1, x_2, \dots)$ is a point in X , this corresponds to the sequence $\{(x_1, \dots, x_n)\}_{n \in \mathbb{N}} \in \prod Y_n$ and by Proposition 1.7 we have

$$A_x = \varinjlim (B_n)_{(x_1, \dots, x_n)}.$$

Using Proposition 1.6 we obtain

$$A_x = \varinjlim \bigotimes_{k=1}^n (A_k)_{x_k} = \bigotimes_{n=1}^{\infty} (A_n)_{x_n}.$$

□

1.9 Let A be a $\mathcal{C}_0(X)$ -algebra. Since X is locally compact and σ -compact there is an increasing sequence $\{V_i\}_{i \in \mathbb{N}}$ of open subsets of X such that each V_i has compact closure K_i and $X = \bigcup_{i \in \mathbb{N}} V_i$. Each $\mathcal{C}_0(V_i) \cdot A$ is an ideal of A_{K_i} which, at the same time, is a $\mathcal{C}_0(V_i)$ -algebra (with the obvious action of $\mathcal{C}_0(V_i)$). We have $\mathcal{C}_0(V_i) \cdot A \subset \mathcal{C}_0(V_{i+1}) \cdot A$ for all i and $A = \varinjlim \mathcal{C}_0(V_i) \cdot A$. This in particular shows:

PROPOSITION: *Let \mathcal{I} be a property of (separable) C^* -algebras which passes to ideals and inductive limits. Let X be a locally compact σ -compact space and suppose that, for any compact subset K of X and any (separable) $\mathcal{C}(K)$ -algebra B one can show that if each B_x , $x \in K$, satisfies \mathcal{I} , then so does B . It follows that if A is a (separable) $\mathcal{C}_0(X)$ -algebra such that each A_x , $x \in X$, satisfies \mathcal{I} , then so does A .*

Property \mathcal{I} in the preceding proposition could for example be stability or \mathcal{D} -stability (where \mathcal{D} is a K_1 -injective strongly self-absorbing C^* -algebra, cf. [32, Corollary 2.3] and [34, Corollaries 3.1 and 3.4]).

1.10 Suppose X is a compact subset of a compact space Y , and suppose that A is a $\mathcal{C}(X)$ -algebra. The restriction map $\mathcal{C}(Y) \rightarrow \mathcal{C}(X)$, composed with the structure map from $\mathcal{C}(X)$, gives A the structure of a $\mathcal{C}(Y)$ -algebra. If $x \in X \subseteq Y$, then the fibre A_x is the same regardless of whether A is viewed as a $\mathcal{C}(X)$ -algebra or a $\mathcal{C}(Y)$ -algebra. If $x \in Y \setminus X$, then $A_x = 0$.

2. ALMOST UNPERFORATED CUNTZ SEMIGROUP

2.1 We remind the reader about the ordered Cuntz semigroup $W(A)$ associated to a C^* -algebra A from [7] (see also [33]). Let $M_\infty(A)^+$ denote the (disjoint) union $\bigcup_{n=1}^\infty M_n(A)^+$. For $a \in M_n(A)^+$ and $b \in M_m(A)^+$ set $a \oplus b = \text{diag}(a, b) \in M_{n+m}(A)^+$, and write $a \precsim b$ if there is a sequence $\{x_k\}$ in $M_{m,n}(A)$ such that $x_k^* b x_k \rightarrow a$. Write $a \sim b$ if $a \precsim b$ and $b \precsim a$. Put $W(A) = M_\infty(A)^+ / \sim$, and let $\langle a \rangle \in W(A)$ be the equivalence class containing a . Then $W(A)$ is an ordered abelian semigroup when equipped with the relations:

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle, \quad \langle a \rangle \leq \langle b \rangle \iff a \precsim b, \quad a, b \in M_\infty(A)^+.$$

Any $*$ -homomorphism $\varphi: A \rightarrow B$ between C^* -algebras A and B induces a morphism $\varphi_*: W(A) \rightarrow W(B)$ by $\varphi_*(\langle a \rangle) = \langle \varphi_n(a) \rangle$, when a is a positive element in $M_n(A)$ and $\varphi_n: M_n(A) \rightarrow M_n(B)$ is the natural extension of φ . It is easy to check that φ_* is additive and order preserving.

If A_0 is a hereditary sub- C^* -algebra of A and if $\iota: A_0 \rightarrow A$ is the inclusion mapping, then $\iota_*: W(A_0) \rightarrow W(A)$ is an order-isomorphism (from $W(A_0)$ onto $\iota_*(W(A_0))$). (This is to say that ι_* is injective, and that for $x, y \in W(A_0)$ one has $\iota_*(x) \leq \iota_*(y)$ if and only if $x \leq y$.) We can therefore suppress ι_* and identify $W(A_0)$ with a sub-semigroup of $W(A)$.

2.2 Recall (from [33]) that an ordered abelian semigroup $(W, +, \leq)$ is said to be *almost unperforated* if, whenever $x, y \in W$ and $n, m \in \mathbb{N}$ are such that $nx \leq my$ and $n > m$, one has $x \leq y$. Equivalently, W is almost unperforated if $(n+1)x \leq ny$ implies $x \leq y$.

We shall write $(a - \varepsilon)_+$ for $f_\varepsilon(a)$, when a is a positive element in A and when $f_\varepsilon(t) = \max\{t - \varepsilon, 0\}$. If $a \precsim b$, then for every $\varepsilon > 0$ there exists $t \in A$ such that $(a - \varepsilon)_+ = t^* b t$. If a, b are positive elements in A with $\|a - b\| < \varepsilon$, then $(a - \varepsilon)_+ \precsim b$. (See eg. [27] for this.)

2.3 PROPOSITION: *Let $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ be a short exact sequence of separable C^* -algebras. If $W(A)$ is almost unperforated, then so are $W(I)$ and $W(A/I)$.*

PROOF: Following the remarks above, we identify $W(I)$ with its image in $W(A)$. Being almost unperforated clearly passes to sub-semigroups (with the inherited order), so $W(I)$ is almost unperforated.

We proceed to show that $W(A/I)$ is almost unperforated. Let $x, y \in W(A/I)$ and $n \in \mathbb{N}$ be given such that $(n+1)x \leq ny$. We must show that $x \leq y$. Let $\pi: A \rightarrow A/I$ be the quotient mapping. Since positive elements in (matrix algebras over) A/I lift to positive elements in (matrix algebras over) A , upon replacing A with a suitable matrix algebra over A , we can assume that there are positive elements a, b in A such that $x = \langle \pi(a) \rangle$ and $y = \langle \pi(b) \rangle$. Let $\varepsilon > 0$ be given. Let $c \otimes 1_k$ denote the k -fold direct sum $c \oplus c \oplus \dots \oplus c$. Then, as $\pi(a) \otimes 1_{n+1} \precsim \pi(b) \otimes 1_n$, there exists an element s in $M_{n,n+1}(A)$ such that

$$\pi(s)^*(\pi(b) \otimes 1_n)\pi(s) = (\pi(a) - \varepsilon)_+ \otimes 1_{n+1} = \pi((a - \varepsilon)_+) \otimes 1_{n+1}.$$

Put $c = s^*(b \otimes 1_n)s - (a - \varepsilon)_+ \otimes 1_{n+1}$. Then c belongs to $M_{n+1}(I)$ and

$$\begin{aligned} (a - \varepsilon)_+ \otimes 1_{n+1} &= s^*(b \otimes 1_n)s - c \leq s^*(b \otimes 1_n)s + |c| \\ &\lesssim (b \otimes 1_n) \oplus |c|. \end{aligned}$$

In other words, $(n+1)\langle (a - \varepsilon)_+ \rangle \leq n\langle b \rangle + \langle |c| \rangle \leq n(\langle b \rangle + \langle |c| \rangle)$, whence $\langle (a - \varepsilon)_+ \rangle \leq \langle b \rangle + \langle |c| \rangle$ because $W(A)$ is assumed to be almost unperforated. Applying π_* to the last inequality yields $\langle (\pi(a) - \varepsilon)_+ \rangle \leq \langle \pi(b) \rangle$. Finally, as $\varepsilon > 0$ was arbitrary we get the desired inequality: $x = \langle \pi(a) \rangle \leq \langle \pi(b) \rangle = y$. \square

3. STABILITY

We show here that a $\mathcal{C}_0(X)$ -algebra A is stable if and only if all its fibres A_x are stable provided that either X is finite dimensional or A is \mathcal{Z} -stable. It is not true in general that $\mathcal{C}_0(X)$ -algebras with stable fibres are stable, as shown in Example 3.11 (which essentially is taken from [28]). Enroute we give a general characterization of when a separable C^* -algebra with almost unperforated Cuntz semigroup is stable (Theorem 3.6). Furthermore, we show that stability passes to extensions if the algebra in the middle has almost unperforated Cuntz semigroup.

3.1 In [15, Theorem 2.1] various (algebraic) characterizations were given of when a separable C^* -algebra A is stable. One of the equivalent conditions is as follows: for any $a \in F(A)$ there is $b \in F(A)$ such that $a \perp b$ and $a \lesssim b$. Here $F(A)$ is the set of positive elements a in A for which there exists a positive element e in A with $a = ae = ea$.

3.2 LEMMA: *Let A be a separable C^* -algebra.*

- (i) *Let $a \in F(A)$ be given. Then there is an approximate unit $\{e_n\}_{n=1}^\infty$ for A such that $e_1 a = a$ and $e_{n+1} e_n = e_n$ for all n .*
- (ii) *Suppose that A is stable and that $\{e_n\}_{n=1}^\infty$ is an approximate unit for A satisfying $e_{n+1} e_n = e_n$ for all n . Then, for each $k \in \mathbb{N}$ there exists $\ell > k$ such that $e_1 \lesssim e_\ell - e_k$.*

PROOF: (i). Take a positive contraction $f \in A$ such that $fa = a$. As $\overline{(1-f)A(1-f)}$ is a separable C^* -algebra, it contains a strictly positive element d . Notice that $d \perp a$. It is easy to verify that $h := f + d$ is strictly positive in A . Note that $\varphi(h)a = a$ whenever φ is a continuous function from \mathbb{R}^+ into itself with $\varphi(0) = 0$ and $\varphi(1) = 1$. Indeed, by Weierstrass' theorem it suffices to show this for polynomials, and by linearity it suffices to consider the case where $\varphi(t) = t^k$, i.e., we must show that $(f + d)^k a = a$. The expression on the left-hand side expands in 2^k terms, the first of which is $f^k a$ (which is a), and the remaining $2^k - 1$ terms are of the form $wdf^\ell a$, where $\ell \geq 0$ and where w is a word in f and d of length $k - 1 - \ell$. As $wdf^\ell a = wda = 0$ the claim is proved.

Take now a sequence $\{\varphi_n\}_{n=1}^\infty$ of continuous functions $\varphi_n: \mathbb{R}^+ \rightarrow [0, 1]$ such that φ_n is zero on $[0, 1/(n+1)]$, linear on $[1/(n+1), 1/n]$, and equal to 1 on $[1/n, \infty)$. Then the sequence $\{e_n\}$ defined by $e_n = \varphi_n(h)$ has the desired properties.

(ii). As e_1 belongs to $\overline{(e_2 - 1/2)_+ A (e_2 - 1/2)_+}$, we have $e_1 \lesssim (e_2 - 1/2)_+$. Assume without loss of generality that $k \geq 2$. By stability of A , and because e_k belongs to $F(A)$, there exists a positive element $c \in A$ such that $c \perp e_k$ and $e_2 \lesssim e_k \lesssim c$. For each $\ell > k$ put $c_\ell := e_\ell c e_\ell = (e_\ell - e_k)c(e_\ell - e_k) \lesssim e_\ell - e_k$. Then

$c_\ell \rightarrow c$. Find $\eta > 0$ such that $(e_2 - 1/2)_+ \preceq (c - \eta)_+$, and find $\ell > k$ such that $\|c - c_\ell\| < \eta$. Then

$$e_1 \preceq (e_2 - 1/2)_+ \preceq (c - \eta)_+ \preceq c_\ell \preceq e_\ell - e_k.$$

□

3.3 LEMMA: *Let X be a compact Hausdorff space and let A be a separable $\mathcal{C}(X)$ -algebra. Suppose that A_x is stable for all $x \in X$. Then, for each $a \in F(A)$, there is a sequence a_1, a_2, a_3, \dots of positive elements in A such that a, a_1, a_2, \dots are pairwise orthogonal and such that $\pi_x(a) \preceq \pi_x(a_j)$ for all $x \in X$ and for all j .*

PROOF: Let $a \in F(A)$ be given. Choose an increasing approximate unit $\{e_n\}$ for A consisting of positive contractions for which $e_1 a = a = a e_1$ and $e_{n+1} e_n = e_n$ for all n (cf. Lemma 3.2). We show that for each k there is $\ell > k$ such that $\pi_x(a) \preceq \pi_x(e_\ell - e_k)$ for all $x \in X$. Let $k \in \mathbb{N}$ be fixed.

By stability of A_x there is $\ell_x > k$ such that $\pi_x(e_1) \preceq \pi_x(e_{\ell_x} - e_k)$ (cf. Lemma 3.2). Take $t_x \in A$ such that

$$\|\pi_x(t_x^*(e_{\ell_x} - e_k)t_x) - \pi_x(e_1)\| < 1/2.$$

Let W_x be the open neighborhood of x consisting of all points $y \in X$ for which

$$\|\pi_y(t_x^*(e_{\ell_x} - e_k)t_x) - \pi_y(e_1)\| < 1/2.$$

It follows from the relation $a e_1 = a$ that a belongs to $\overline{(e_1 - 1/2)_+ A (e_1 - 1/2)_+}$, and in particular that $a \preceq (e_1 - 1/2)_+$. Hence

$$\pi_y(a) \preceq \pi_y((e_1 - 1/2)_+) \preceq \pi_y(e_{\ell_x} - e_k) \preceq \pi_y(e_\ell - e_k)$$

for all $y \in W_x$ and for all $\ell \geq \ell_x$. Refine the open cover $\{W_x\}_{x \in X}$ to a finite open cover $\{W_x\}_{x \in F}$. Put $\ell = \max\{\ell_x \mid x \in F\}$. Then $\pi_x(a) \preceq \pi_x(e_\ell - e_k)$ for all $x \in X$.

We can now take a_j to be $e_{k_j} - e_{k_{j-1}+1}$, where $2 = k_0 < k_1 < k_2 < \dots$ is a sequence of natural numbers chosen such that $\pi_x(a) \preceq \pi_x(e_{k_j} - e_{k_{j-1}+1})$ for all $x \in X$ and all j . □

3.4 PROPOSITION: *Let X be a locally compact metrizable space of finite covering dimension. Let A be a separable $C_0(X)$ -algebra. Then, A is stable if and only if A_x is stable for all $x \in X$.*

PROOF: Since stability passes to quotients by [32, Corollary 2.3(ii)], each A_x is stable if A is. We show the converse. By 1.9, it suffices to prove the assertion for compact X of finite covering dimension.

Let a be an element in $F(A)$ and let $\varepsilon > 0$ be given. We show that there is an element $t \in A$ such that $\|a - t^*t\| \leq \varepsilon$ and $tt^* \perp a$. This will show that A is stable (by [15, Theorem 2.1 and Proposition 2.2]). Denote the dimension of the space X by n .

It follows from Lemma 3.3 that there are positive elements a_1, a_2, \dots, a_{n+1} in A satisfying $\pi_x(a) \preceq \pi_x(a_j)$ for all $x \in X$ and for all j , and such that $a, a_1, a_2, \dots, a_{n+1}$ are pairwise orthogonal.

For each $x \in X$ there are elements $s_{j,x}$ in A such that $\|\pi_x(s_{j,x}^* a_j s_{j,x} - a)\| < \varepsilon$ for $j = 1, 2, \dots, n+1$. Put $t_{j,x} := a_j^{1/2} s_{j,x}$. Then

$$\|\pi_x(t_{j,x}^* t_{j,x} - a)\| < \varepsilon, \quad t_{j,x} t_{j,x}^* \in \overline{a_j A a_j}.$$

For each $x \in X$, let U_x be the open neighborhood of x consisting of all $y \in X$ for which $\|\pi_y(t_{j,x}^* t_{j,x} - a)\| < \varepsilon$ for all $j = 1, 2, \dots, n+1$.

Because X has dimension n , the open cover $\{U_x\}_{x \in X}$ of X has an open subcover $\{V_{j,\alpha}\}$, with $j = 1, 2, \dots, n+1$ and with α in some (finite) index set I_j , such that the sets $\{V_{j,\alpha}\}_{\alpha \in I_j}$ are pairwise disjoint for each fixed j . Relabeling the elements $t_{j,x}$ we get elements $t_{j,i,\alpha}$ in A such that

$$\|\pi_y(t_{j,i,\alpha}^* t_{j,i,\alpha} - a)\| < \varepsilon, \quad t_{j,i,\alpha} t_{j,i,\alpha}^* \in \overline{a_j A a_j},$$

for all $i, j = 1, 2, \dots, n+1$, for all $\alpha \in I_i$, and for all $y \in V_{i,\alpha}$. Let $\{\varphi_{i,\alpha}\}$ be a partition of the unit subordinate to the cover $\{V_{i,\alpha}\}$, i.e., $\sum_{i,\alpha} \varphi_{i,\alpha} = 1$ and $\text{supp}(\varphi_{i,\alpha}) \subseteq V_{i,\alpha}$. Put

$$t := \sum_{j=1}^{n+1} \sum_{\alpha \in I_j} \varphi_{j,\alpha}^{1/2} t_{j,j,\alpha}.$$

As $at_{j,j,\alpha} = 0 = t_{i,i,\beta}^* a$ for all i, j and α, β , we see that $tt^* \perp a$. Next, using that $t_{j,j,\alpha}^* t_{i,i,\beta} = 0$ if $i \neq j$ and that $\varphi_{j,\alpha}^{1/2} \varphi_{j,\beta}^{1/2} = 0$ if $\alpha \neq \beta$, we get

$$\begin{aligned} t^* t &= \sum_{j,\alpha} \sum_{i,\beta} \varphi_{j,\alpha}^{1/2} \varphi_{i,\beta}^{1/2} t_{j,j,\alpha}^* t_{i,i,\beta} \\ &= \sum_{j,\alpha} \varphi_{j,\alpha} t_{j,j,\alpha}^* t_{j,j,\alpha}. \end{aligned}$$

If $\varphi_{j,\alpha}(x) \neq 0$, then $x \in V_{j,\alpha}$ and $\|\pi_x(t_{j,j,\alpha}^* t_{j,j,\alpha} - a)\| < \varepsilon$. Hence

$$\|\pi_x(\varphi_{j,\alpha}(t_{j,j,\alpha}^* t_{j,j,\alpha} - a))\| = \varphi_{j,\alpha}(x) \|\pi_x(t_{j,j,\alpha}^* t_{j,j,\alpha} - a)\| \leq \varepsilon \varphi_{j,\alpha}(x),$$

for all $x \in X$. We conclude that $\|\pi_x(t^* t - a)\| \leq \varepsilon$ for all $x \in X$, so that

$$\|t^* t - a\| = \sup_{x \in X} \|\pi_x(t^* t - a)\| \leq \varepsilon$$

as desired. \square

3.5 It was shown in [28] that there is a non-stable separable C^* -algebra A such that $M_n(A)$ is stable for some n (see Example 3.11 below). We shall now use Proposition 3.4 to show that this phenomenon cannot occur if A is \mathcal{Z} -stable.

COROLLARY: *Suppose A is a separable, \mathcal{Z} -stable C^* -algebra. If $M_n(A)$ is stable for some n , then so is A .*

Proof. By [28, Proposition 2.1], if $M_n(A)$ is stable, then so is $M_{n+1}(A)$. Denote

$$\mathcal{Z}_{n,n+1} = \{f \in \mathcal{C}([0, 1], M_n \otimes M_{n+1}) \mid f(0) \in M_n \otimes 1, f(1) \in 1 \otimes M_{n+1}\}.$$

We know that there is a unital embedding ι of $\mathcal{Z}_{n,n+1}$ into \mathcal{Z} (see [18]). Consider the inductive system

$$A \otimes \mathcal{Z}_{n,n+1} \xrightarrow{id \otimes \iota} A \otimes \mathcal{Z} \xrightarrow{x \mapsto x \otimes 1} A \otimes \mathcal{Z} \otimes \mathcal{Z}_{n,n+1} \xrightarrow{id \otimes \iota} A \otimes \mathcal{Z} \otimes \mathcal{Z} \xrightarrow{x \mapsto x \otimes 1} \dots$$

The inductive limit is the same as that of

$$A \xrightarrow{x \mapsto x \otimes 1} A \otimes \mathcal{Z} \xrightarrow{x \mapsto x \otimes 1} A \otimes \mathcal{Z} \otimes \mathcal{Z} \xrightarrow{x \mapsto x \otimes 1} \dots$$

which is isomorphic to A . By skipping all the even places in the first diagram, and noting that $A \cong A \otimes \mathcal{Z}$, we see that A can be written as an inductive limit of algebras of the form $A \otimes \mathcal{Z}_{n,n+1}$. We view $A \otimes \mathcal{Z}_{n,n+1}$ as a $\mathcal{C}([0, 1])$ -algebra, by

taking the embedding of $\mathcal{C}([0, 1])$ into $\mathcal{Z}_{n,n+1}$ as scalar functions. The fibres are $A \otimes M_n$ at 0, $A \otimes M_{n+1}$ at 1, and $A \otimes M_n \otimes M_{n+1}$ elsewhere. In particular, the fibres are all stable. By Proposition 3.4, $A \otimes \mathcal{Z}_{n,n+1}$ is stable. Since stability passes to inductive limits, it follows that A is stable, as required. \square

In Corollary 3.12 we shall give an alternative proof of Corollary 3.5, using Theorem 3.6 rather than Proposition 3.4.

3.6 We give below a characterization of stability for separable C^* -algebras whose Cuntz semigroup of positive elements is almost unperforated. This result is very similar to [15, Proposition 5.1]. We refer to [2] for the definition and properties of quasi-traces and 2-quasi-traces.

THEOREM: *Let A be a separable C^* -algebra for which $W(A)$ is almost unperforated. Then A is stable if and only if A has no bounded non-zero 2-quasi-trace and A has no non-zero unital quotient.*

PROOF: The “only if” part holds without the assumption that $W(A)$ is almost unperforated: If A is stable, then so is any non-zero quotient of A , and no (non-zero) unital C^* -algebra is stable. Let now τ be a non-zero 2-quasi-trace on A . Then $\tau(a) > 0$ for some $a \in F(A)$. By stability there is a sequence $\{a_n\}_{n=1}^\infty$ of pairwise orthogonal elements in A each equivalent to a , and hence with $\tau(a_n) = \tau(a)$ for all n . (Two positive elements a and b in a C^* -algebra are *equivalent* if $a = x^*x$ and $b = xx^*$ for some x in the C^* -algebra.) But then τ cannot be bounded.

To prove the “if” part, we show that for each $a \in F(A)$ there is $b \in A^+$ such that $a \perp b$ and $a \precsim b$ (cf. [15, Theorem 2.1 and Proposition 2.2]). Accordingly, take a in $F(A)$, and let B be the hereditary sub- C^* -algebra of A consisting of all $x \in A$ for which $xx^* \perp a$ and $x^*x \perp a$. We find a positive element $b \in B$ such that $a \precsim b$.

By a *state on $W(A)$ normalized at $x \in W(A)$* we mean an additive order preserving map from $W(A)$ into $\mathbb{R}^+ \cup \{\infty\}$ that maps x to 1. The set of all such states is denoted by $S(W, x)$. By [33, Proposition 3.2] (essentially an argument from [12]) and the assumption that $W(A)$ is almost unperforated, we can conclude that $a \precsim b$ (or, equivalently, $\langle a \rangle \leq \langle b \rangle$) if $\langle a \rangle \leq N\langle b \rangle$ for some N and if $f(\langle a \rangle) < f(\langle b \rangle)$ for all $f \in S(W(A), \langle b \rangle)$.

Since $a \in F(A)$ we can find $e, e' \in F(A)$ such that $ae = a$ and $ee' = e$.

We remark that B is full in A . Suppose, to reach a contradiction, that the closed two-sided ideal I generated by B is proper. Then $e + I$ is a unit for A/I contrary to our assumptions. (Indeed, for all $x \in A$ we have $ex + I = x + I = xe + I$. To see the former identity, put $y = ex - x$ and note that $yy^* \perp a$, so yy^* belongs to B , whence y belongs to I .)

We next show that B contains an element b_0 such that $\langle e \rangle \leq N\langle b_0 \rangle$ for some N . The set $F(A)$ is contained in the Pedersen ideal of A , so e belongs to the algebraic ideal generated by B . Hence there exist b_1, \dots, b_N in B^+ and x_1, \dots, x_N in A such that $e \leq \sum_{j=1}^N x_j^* b_j x_j$. Put $b_0 = \sum_{j=1}^N b_j$. Then $\langle b_j \rangle \leq \langle b_0 \rangle$ for all j , so $\langle e \rangle \leq \sum_{j=1}^N \langle b_j \rangle \leq N\langle b_0 \rangle$.

We now show that

$$(2) \quad \sup\{f(\langle b \rangle) \mid b \in B^+\} = \infty$$

for all $f \in S(W(A), \langle e \rangle)$. Suppose, to reach a contradiction, that $f_0 \in S(W(A), \langle e \rangle)$ is such that the supremum in (2) is finite, say equal to C . Then $f_0(\langle x \rangle) \leq C + 1$ for all $x \in A^+$. Indeed,

$$\begin{aligned} x &= x^{1/2}ex^{1/2} + x^{1/2}(1-e)x^{1/2} \lesssim x^{1/2}ex^{1/2} \oplus x^{1/2}(1-e)x^{1/2} \\ &\sim e^{1/2}xe^{1/2} \oplus (1-e)^{1/2}x(1-e)^{1/2} \lesssim e \oplus (1-e)x(1-e), \end{aligned}$$

cf. [24, Lemma 2.8], so $f_0(\langle x \rangle) \leq f_0(\langle e \rangle) + C = 1 + C$, because $(1-e)x(1-e)$ belongs to B .

Let \bar{f}_0 be the lower semicontinuous dimension function arising from f_0 , i.e.,

$$\bar{f}_0(\langle c \rangle) = \lim_{\varepsilon \rightarrow 0^+} f_0(\langle (c - \varepsilon)_+ \rangle), \quad c \in M_\infty(A).$$

Then there is a 2-quasi-trace τ on A such that

$$\bar{f}_0(\langle c \rangle) = \lim_{n \rightarrow \infty} \tau(c^{1/n})$$

for $c \in M_\infty(A)$. To reach a contradiction with our assumptions we show that τ is non-zero and bounded. To see the former, note that $1 = f_0(\langle e \rangle) \leq \bar{f}_0(\langle e' \rangle) \leq f_0(\langle e' \rangle) \leq C + 1 < \infty$. The latter follows from the formula

$$\tau(c) = \int_0^{\|c\|} \bar{f}_0(\langle (c - t)_+ \rangle) dt, \quad c \in A^+,$$

which shows that $\|\tau\| \leq C + 1$. This completes the proof of (2).

The set $S(W(A), \langle e \rangle)$ is compact when equipped with the topology of point-wise convergence (the weak*-topology). We can therefore find b'_1, \dots, b'_n in B^+ such that for each $f \in S(W(A), \langle e \rangle)$ there is at least one j for which $f(\langle b'_j \rangle) > 2$. Put $b = b_0 + b'_1 + \dots + b'_n$. Then $\langle a \rangle \leq \langle e \rangle \leq N\langle b_0 \rangle \leq N\langle b \rangle$; and $\langle b'_j \rangle \leq \langle b \rangle$ for all j , so $f(\langle b \rangle) \geq 2$ for all $f \in S(W(A), \langle e \rangle)$.

To complete the proof we must show that $f(\langle a \rangle) < f(\langle b \rangle)$ for all $f \in S(W(A), \langle b \rangle)$. Take such a state f , and note that $f(\langle e \rangle) \leq Nf(\langle b \rangle) = N < \infty$. If $f(\langle e \rangle) = 0$, then $f(\langle a \rangle) = 0 < 1 = f(\langle b \rangle)$, because $a \lesssim e$. And if $f(\langle e \rangle) = \alpha > 0$, then $\alpha^{-1}f$ belongs to $S(W(A), \langle e \rangle)$, in which case we have

$$\alpha^{-1}f(\langle a \rangle) \leq \alpha^{-1}f(\langle e \rangle) = 1 < 2 < \alpha^{-1}f(\langle b \rangle),$$

as desired. \square

3.7 The proposition below is contained in Haagerup's manuscript [13]. We restate it here and give a short proof based on a result from [2].

PROPOSITION: *Any 2-quasi-trace τ defined on a C^* -algebra A , which vanishes on a closed two-sided ideal I in A , factors through the quotient A/I , i.e., there is a 2-quasi-trace $\bar{\tau}$ on A/I such that $\tau = \bar{\tau} \circ \pi$, where $\pi: A \rightarrow A/I$ is the quotient mapping.*

PROOF: We wish to define $\bar{\tau}$ by $\bar{\tau}(a + I) = \tau(a)$ for $a \in A$; and we must check that this is well-defined, i.e., we must show that $\tau(a + x) = \tau(a)$ for all $a \in A$ and for all $x \in I$. Let $\{e_\alpha\}$ be an increasing approximate unit for I consisting of positive contractions, and such that $\{e_\alpha\}$ is asymptotically central for A . As τ is continuous (cf. [2, Corollary II.2.5]) it suffices to show that $\tau((1 - e_\alpha)a(1 - e_\alpha)) \rightarrow \tau(a)$ for all $a \in A$ (because $\|(1 - e_\alpha)(a + x)(1 - e_\alpha) - (1 - e_\alpha)a(1 - e_\alpha)\| \rightarrow 0$). As quasi-traces by definition are self-adjoint, it suffices to show this for self-adjoint elements $a \in A$.

Fixing such an element a , put $b_\alpha = (1 - e_\alpha)a(1 - e_\alpha)$ and $x_\alpha = a - b_\alpha$. Then x_α belongs to I , whence

$$\begin{aligned} |\tau(a) - \tau(b_\alpha)| &= |\tau(a) - \tau(a - x_\alpha)| \\ &\leq |\tau(a) - \tau(a - x_\alpha) - \tau(x_\alpha)| + |\tau(x_\alpha)| \\ &= |\tau(a) - \tau(a - x_\alpha) - \tau(x_\alpha)| \rightarrow 0, \end{aligned}$$

by [2, Corollary II.2.6] because $\|ax_\alpha - x_\alpha a\| \rightarrow 0$. \square

3.8 The proposition below is due to Uffe Haagerup (private communication).

PROPOSITION: *Let X be a compact Hausdorff space (not necessarily of finite dimension) and let A be a separable $\mathcal{C}(X)$ -algebra. If A admits a bounded non-zero 2-quasi-trace, then so does A_x for some x in X .*

PROOF: Let $QT(A)$ be the compact convex set of all 2-quasi-traces on A of norm 1, and suppose that $QT(A)$ is non-empty. Let τ be an extreme point in $QT(A)$. We show that there is a functional ρ on $\mathcal{C}(X)$ such that $\tau(fa) = \rho(f)\tau(a)$ for all $f \in \mathcal{C}(X)^+$ and for all $a \in A$.

To this end let \mathcal{M} denote the set of positive contractions in $\mathcal{C}(X)$, and fix an element f in \mathcal{M} . Put $\tau_1(a) = \tau(fa)$ and $\tau_2(a) = \tau((1 - f)a)$. As fa and $(1 - f)a$ commute for all (self-adjoint) a we see that $\tau = \tau_1 + \tau_2$ (this identity is first verified on self-adjoint elements, and then extended to all elements in A using that τ is self-adjoint). The norm of a 2-quasi-trace σ on A is equal to $\sup_n \sigma(e_n)$, where $\{e_n\}$ is any increasing approximate unit for A consisting of positive contractions. In particular, $\|\tau\| = \|\tau_1\| + \|\tau_2\|$. By extremality of τ we conclude that τ_1 is proportional to τ , and so there exists a constant $\rho(f)$ such that $\tau_1(a) = \rho(f)\tau(a)$ for all $a \in A$. This shows that $\tau(fa) = \rho(f)\tau(a)$ for all $f \in \mathcal{M}$ and for all $a \in A$.

Let $a \in A^+$ with $\tau(a) > 0$ be given. Put $\rho_a(f) = \tau(fa)/\tau(a)$. Then $\rho_a: \mathcal{C}(X) \rightarrow \mathbb{C}$ is linear and $\rho_a(f) = \rho(f)$ for all $f \in \mathcal{M}$. As a linear functional on $\mathcal{C}(X)$ is determined by its values on \mathcal{M} it follows that ρ_a is independent of the choice of $a \in A^+$, and that ρ extends uniquely to a linear functional on $\mathcal{C}(X)$, again denoted by ρ , such that $\tau(fa) = \rho(f)\tau(a)$ for all $f \in \mathcal{C}(X)$ and all $a \in A^+$ with $\tau(a) > 0$. If $a \in A^+$ and $\tau(a) = 0$, then $\tau(fa) = 0 = \rho(f)\tau(a)$ for all $f \in \mathcal{C}(X)$. In conclusion we have $\tau(fa) = \rho(f)\tau(a)$ whenever $f \in \mathcal{C}(X)$ and $a \in A^+$, or (since τ is self-adjoint and additive on commuting self-adjoint elements) for all $f \in \mathcal{C}(X)^+$ and for all $a \in A$.

Fix a positive element a with $\tau(a) > 0$, and let $f, g \in \mathcal{C}(X)$. Then

$$\rho(fg)\tau(a) = \tau(fga) = \rho(f)\tau(ga) = \rho(f)\rho(g)\tau(a).$$

This shows that ρ is multiplicative. It follows that there exists $x \in X$ such that $\rho(f) = f(x)$ for all $f \in \mathcal{C}(X)$.

We can now conclude that τ vanishes on $I_x = \mathcal{C}_0(X \setminus \{x\}) \cdot A$. It follows from Proposition 3.7 that τ drops to the quotient A_x , i.e., that there is a 2-quasi-trace $\bar{\tau}$ on A_x such that $\tau = \bar{\tau} \circ \pi_x$. In particular, $\bar{\tau}$ is a bounded non-zero 2-quasi-trace on A_x . \square

3.9 **PROPOSITION:** *Let X be a locally compact Hausdorff space (not necessarily of finite dimension) and let A be a separable $\mathcal{C}_0(X)$ -algebra. Suppose that $W(A)$ is*

almost unperforated (for example, A could be \mathcal{Z} -stable). Then, A is stable if and only if A_x is stable for each $x \in X$.

PROOF: As in the proof of Proposition 3.4, it is clear that stability passes from A to each fibre A_x . Again by 1.9 (and using Proposition 2.3), it suffices to prove the converse for compact X .

By Theorem 3.6 it suffices to show that A has no non-zero bounded 2-quasi-trace and that A has no non-zero unital quotient. The former follows from Proposition 3.8 since no fibre A_x admits a non-zero bounded 2-quasi-trace (again by Theorem 3.6).

Suppose that J is a closed two-sided ideal in A such that A/J is unital. Then $A_x/\pi_x(J)$ is unital and at the same time stable (being a quotient of A/J , cf. [32, Corollary 2.3(ii)]), and is therefore necessarily zero (by Theorem 3.6). This proves that $\pi_x(J) = A_x$ for all x , whence $J = A$. To see the latter, note that if $J \neq A$, then there is a positive contraction $a \in A$ such that $\|a + J\| = 1$. Let $\{e_n\}$ be an approximate unit for J . Then $\|(1 - e_n)a(1 - e_n)\| = 1$ for all n . So there is $x_n \in X$ with $\|\pi_{x_n}((1 - e_n)a(1 - e_n))\| \geq 1/2$. Let x_0 be an accumulation point for $\{x_n\}$. Then $\|\pi_{x_0}((1 - e_n)a(1 - e_n))\| \geq 1/2$ for all n . But then $\pi_{x_0}(a) \notin \pi_{x_0}(J)$. \square

3.10 COROLLARY: *Let $0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0$ be a short exact sequence of separable C^* -algebras.*

- (i) *If J and B are both stable and $W(A)$ is almost unperforated (for example, A could be \mathcal{Z} -stable), then A is stable.*
- (ii) *If both J and B are stable and \mathcal{Z} -stable, then A is also stable and \mathcal{Z} -stable.*

PROOF: (i). In view of Theorem 3.6 we have to show that A has no bounded non-zero 2-quasi-trace and no non-zero unital quotient. Let $\pi: A \rightarrow B$ denote the quotient mapping.

Suppose, to reach a contradiction, that I is a proper closed two-sided ideal in A and that A/I is unital. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & A & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J/(J \cap I) & \longrightarrow & A/I & \longrightarrow & B/\pi(I) & \longrightarrow & 0. \end{array}$$

We have either $\pi(I) = B$ or $B/\pi(I)$ is unital (and non-zero). The latter is impossible because B is stable. Hence $\pi(I) = B$, in which case $J/(J \cap I)$ is isomorphic to A/I . The latter is unital and non-zero; hence $J/(J \cap I)$ is unital and non-zero, contradicting that J is stable.

Suppose next that τ is a bounded 2-quasi-trace on A . The restriction of τ to J is also a bounded 2-quasitrace, and is hence zero by Theorem 3.6. By Proposition 3.7, τ drops to a 2-quasi-trace $\bar{\tau}$ on B . Again by Theorem 3.6 we conclude that $\bar{\tau} = 0$. This entails that $\tau = \bar{\tau} \circ \pi = 0$.

As A has no (non-trivial) unital quotient and no non-zero bounded 2-quasi-trace, Theorem 3.6 yields that A is stable.

(ii). By [34, Theorem 4.3], \mathcal{Z} -stability passes to extensions, whence A is \mathcal{Z} -stable. Now, $W(A)$ is almost unperforated by [33, Theorem 4.5], and (i) applies. \square

It should be noted that not all extensions of stable (separable) C^* -algebras are stable, cf. [30].

3.11 EXAMPLE: We mention here an example from [28, Section 4] of a non-stable $\mathcal{C}(X)$ -algebra B whose fibres B_x are isomorphic to \mathcal{K} for all $x \in X$. This example shows that Propositions 3.4 and 3.9 cannot be improved by removing the condition that X is of finite dimension in the former or that the algebra is \mathcal{Z} -stable in the latter.

In the example X is an infinite cartesian product of Moore spaces Y_n , where n can be taken to be any integer ≥ 2 . The C^* -algebra B is the hereditary sub- C^* -algebra of $\mathcal{C}(X, \mathcal{K})$,

$$B = \overline{\bigcup_{n=1}^{\infty} (p_1 \oplus p_2 \oplus \cdots \oplus p_n) \mathcal{C}(X, \mathcal{K}) (p_1 \oplus p_2 \oplus \cdots \oplus p_n)},$$

where $\{p_j\}$ is a certain sequence of 1-dimensional projections in $\mathcal{C}(X, \mathcal{K})$. The claim to fame of B in the context of [28] is that $M_k(B)$ is non-stable for $1 \leq k < n$, but $M_n(B)$ is stable.

Any hereditary sub- C^* -algebra B of $\mathcal{C}(X, \mathcal{K})$ is a $\mathcal{C}(X)$ -algebra. The fibre map $\pi_x: B \rightarrow B_x$ coincides with the restriction of the evaluation mapping at x to B . Hence $B_x = \pi_x(B) \subseteq \mathcal{K}$. In the case at hand, $B_x \cong \mathcal{K}$ for all $x \in X$ (because B_x is an infinite dimensional hereditary sub- C^* -algebra of \mathcal{K}). Hence all fibres of B are stable, but B itself is not stable.

3.12 COROLLARY: *Suppose A is a separable C^* -algebra such $W(A)$ is almost unperforated (for example, A could be \mathcal{Z} -stable). If $M_n(A)$ is stable for some n , then so is A .*

PROOF: By [2, Proposition II.4.1], a bounded nonzero 2-quasi-trace on A extends to one on $M_n(A)$. If A had a non-zero unital quotient, then clearly so would $M_n(A)$. Thus, since $M_n(A)$ is stable, it follows that A has no unital quotients and does not admit a bounded, non-zero 2-quasi-trace. By Theorem 3.6, A is indeed stable. \square

4. \mathcal{D} -STABILITY

In this section we show that, for a K_1 -injective strongly self-absorbing C^* -algebra \mathcal{D} , a $\mathcal{C}_0(X)$ -algebra A is \mathcal{D} -stable if and only if all its fibres are, provided that X is finite-dimensional (Theorem 4.6). We provide examples (4.7 and 4.8) showing that the above statement can fail with X infinite dimensional, for \mathcal{D} a UHF algebra or the Jiang–Su algebra. However, we show in Proposition 4.11 that if A is ‘locally’ \mathcal{D} -stable then A must be \mathcal{D} -stable (even when X is infinite dimensional).

4.1 To each C^* -algebra A one associates the C^* -algebra $\prod_{\mathbb{N}} A$ of all bounded sequences in A , the C^* -algebra $\bigoplus_{\mathbb{N}} A$ of all sequences in A that converge to zero, and the quotient $A_{\infty} = \prod_{\mathbb{N}} A / \bigoplus_{\mathbb{N}} A$. We view A as embedded in A_{∞} as the (equivalence classes of) constant sequences.

PROPOSITION: *Let A and \mathcal{D} be separable C^* -algebras, such that \mathcal{D} is K_1 -injective and strongly self-absorbing. Then, the following are equivalent:*

- a) A is \mathcal{D} -stable.
- b) Given $\eta > 0$ and finite subsets $\mathcal{F} \subset A$ and $\mathcal{G} \subset \mathcal{D}$, there is a c.p.c. map $\psi: \mathcal{D} \rightarrow A$ such that
 - (i) $\|b\psi(\mathbf{1}_{\mathcal{D}}) - b\| < \eta$
 - (ii) $\|b\psi(d) - \psi(d)b\| < \eta$
 - (iii) $\|b(\psi(dd') - \psi(d)\psi(d'))\| < \eta$
 for all $b \in \mathcal{F}$, $d, d' \in \mathcal{G}$.
- c) Given $\eta > 0$ and a finite subset $\mathcal{F} \subset A$, there are a $*$ -homomorphism $\kappa: A \rightarrow A$ and a unital $*$ -homomorphism $\mu: \mathcal{D} \rightarrow \mathcal{M}(A)$ (the multiplier algebra of A) such that

$$[\kappa(A), \mu(\mathcal{D})] = 0 \text{ and } \|\kappa(b) - b\| < \eta$$

for all $b \in \mathcal{F}$.

- d) There exists a c.p.c. map $\psi: \mathcal{D} \rightarrow A_\infty \cap A'$ such that
 - (i) $b\psi(\mathbf{1}_{\mathcal{D}}) = b$ for all $b \in A$.
 - (ii) $b(\psi(dd') - \psi(d)\psi(d')) = 0$ for all $b \in A$ and all $d, d' \in \mathcal{D}$.

PROOF: a) \Rightarrow c): By [34, Proposition 1.9] there is a sequence of $*$ -homomorphisms $\varphi_n: \mathcal{D} \otimes \mathcal{D} \rightarrow \mathcal{D}$ such that $\varphi_n(d \otimes \mathbf{1}_{\mathcal{D}}) \rightarrow d$ for all $d \in \mathcal{D}$ as $n \rightarrow \infty$. Identify A with $A \otimes \mathcal{D}$. Define a sequence of $*$ -homomorphisms

$$\kappa_n: A \otimes \mathcal{D} \rightarrow A \otimes \mathcal{D}$$

by

$$\kappa_n := (\text{id}_A \otimes \varphi_n) \circ (\text{id}_A \otimes \text{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}).$$

Let A^\sim denote the smallest unitization of A ; it is clear that $A \otimes \mathcal{D}$ is an essential ideal in $A^\sim \otimes \mathcal{D}$, whence the inclusion of $A \otimes \mathcal{D}$ extends to a unital embedding $\iota: A^\sim \otimes \mathcal{D} \hookrightarrow \mathcal{M}(A \otimes \mathcal{D})$. We thus have unital $*$ -homomorphisms

$$\mu_n: \mathcal{D} \rightarrow \mathcal{M}(A \otimes \mathcal{D})$$

given by

$$\mu_n := (\text{id}_{A^\sim} \otimes \varphi_n) \circ (\mathbf{1}_{A^\sim} \otimes \mathbf{1}_{\mathcal{D}} \otimes \text{id}_{\mathcal{D}}).$$

The maps $\kappa := \kappa_{n_0}$ and $\mu := \mu_{n_0}$ for some large enough n_0 will have the right properties.

c) \Rightarrow d): Let $b_1, b_2, \dots \in A$ and $d_1, d_2, \dots \in \mathcal{D}$ be dense sequences in the unit balls of these algebras. Let $\kappa_n: A \rightarrow A$ and $\mu_n: \mathcal{D} \rightarrow \mathcal{M}(A)$ be sequences of homomorphisms which satisfy

$$[\kappa_n(A), \mu_n(\mathcal{D})] = 0 \text{ and } \|\kappa_n(b_k) - b_k\| < 1/n$$

for all $k \leq n$. Using a (possibly uncountable) approximate unit for A which is quasicontral for $\mathcal{M}(A)$, and since $\{\mu_n(d_k) \mid n, k \in \mathbb{N}\}$ is countable, it is routine to find a countable approximate unit h_1, h_2, \dots for A such that $\|b_k h_n - b_k\| < 1/n$ for all $k \leq n$, and such that $\|[\mu_n(d_k), h_n]\| < 1/n$ for all $k \leq n$. Now, define a c.p.c. map $\psi: \mathcal{D} \rightarrow \prod_{\mathbb{N}} A / \bigoplus_{\mathbb{N}} A$ by $d \mapsto \{h_1 \mu_1(d) h_1, h_2 \mu_2(d) h_2, \dots\}$ (i.e., the equivalence class of this sequence), and it is easy to check that ψ has the desired properties.

d) \Rightarrow b): Choose a c.p.c. map $\psi: \mathcal{D} \rightarrow A_\infty \cap A'$. Use the Choi–Effros Theorem to pick some c.p.c. lift $\tilde{\psi}: \mathcal{D} \rightarrow \prod_{\mathbb{N}} A$. Denote by ψ_n the n 'th component of this map (i.e. the composition of the projection onto the n 'th summand of $\prod_{\mathbb{N}} A$ with $\tilde{\psi}$). It is easy to verify now that for a sufficiently large n , $\psi_n: \mathcal{D} \rightarrow A$ will satisfy the required conditions.

b) \Rightarrow a): By separability of A and \mathcal{D} it is straightforward to construct a $*$ -homomorphism $\bar{\psi}: A \otimes \mathcal{D} \rightarrow \prod_{\mathbb{N}} A / \bigoplus_{\mathbb{N}} A$ such that $\bar{\psi}(a \otimes \mathbf{1}_{\mathcal{D}}) = a$ for all $a \in A$. This implies $A \cong A \otimes \mathcal{D}$ by [34, Theorem 2.3]. \square

4.2 To establish \mathcal{D} -stability in Theorem 4.6, we shall mainly make use of characterization b) in Proposition 4.1 above. Most of the work will go into establishing the special case $X = [0, 1]$. To this end, we introduce some ad-hoc terminology. Suppose A is a separable $\mathcal{C}([0, 1])$ -algebra.

DEFINITION: *Suppose we have $\varepsilon > 0$, and finite sets $\mathcal{F} \subset A$, $\mathcal{G} \subset \mathcal{D}$. Suppose $I \subset [0, 1]$ is a closed subset, and $\psi: \mathcal{D} \rightarrow A$ is a c.p.c. map. We shall say that ψ is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for I if it satisfies the following conditions (U), (C) and (M).*

- (U) $\|(b\psi(\mathbf{1}_{\mathcal{D}}) - b)_I\| < \varepsilon$ for all $b \in \mathcal{F}$
- (C) $\|(b\psi(d) - \psi(d)b)_I\| < \varepsilon$ for all $b \in \mathcal{F}$ and $d \in \mathcal{G}$
- (M) $\|(b(\psi(dd') - \psi(d)\psi(d')))_I\| < \varepsilon$ for all $b \in \mathcal{F}$ and $d, d' \in \mathcal{G}$.

Suppose we have some other finite set $\mathcal{G}' \subset \mathcal{D}$ and some $\varepsilon' > 0$. We shall say that ψ is $(\mathcal{F}; \mathcal{G}, \varepsilon; \mathcal{G}', \varepsilon')$ -good for I , where I here is an interval, if ψ is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for I , and there is some closed neighborhood V of the endpoints of I such that ψ is $(\mathcal{F}; \mathcal{G}', \varepsilon')$ -good for V .

The notation “(U),(C),(M)” is intended to serve as mnemonic for (almost) ‘unital’, ‘central’ and ‘multiplicative’, respectively. In our case, we shall have some auxiliary $\mathcal{G} \subseteq \mathcal{G}'$ and $\varepsilon \geq \varepsilon'$, so saying that ψ is $(\mathcal{F}; \mathcal{G}, \varepsilon; \mathcal{G}', \varepsilon')$ -good for I should be thought of as saying that ψ is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for I , and even ‘better’ near the endpoints. Note that Proposition 4.1 b) asserts that A is \mathcal{D} -stable if and only if for any $\varepsilon > 0$ and finite sets $\mathcal{F} \subset A$, $\mathcal{G} \subset \mathcal{D}$, there is a c.p.c. map $\psi: \mathcal{D} \rightarrow A$ which is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[0, 1]$.

4.3 LEMMA: *Let \mathcal{D} be a strongly self-absorbing C^* -algebra. Let A be a separable $\mathcal{C}([0, 1])$ -algebra such that A_x is \mathcal{D} -stable for all $x \in [0, 1]$. Given $\varepsilon > 0$ and finite subsets $\mathcal{F} \subset A$ and $\mathbf{1}_{\mathcal{D}} \in \mathcal{G} \subset \mathcal{D}$, there exist an $n \in \mathbb{N}$, c.p.c. maps $\psi_1, \dots, \psi_n: \mathcal{D} \rightarrow A$, and points $0 = t_0 < t_1 < \dots < t_n = 1$ such that ψ_k is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[t_{k-1}, t_k]$ for $k = 1, \dots, n$.*

PROOF: For any $x \in [0, 1]$ we can find, by Proposition 4.1 b), a c.p.c. map $\sigma_x: \mathcal{D} \rightarrow A_x$ such that for all $b \in \mathcal{F}$, $d, d' \in \mathcal{G}$, we have

- (U) $\|b_x \sigma_x(\mathbf{1}_{\mathcal{D}}) - b_x\| < \varepsilon$
- (C) $\|b_x \sigma_x(d) - \sigma_x(d) b_x\| < \varepsilon$
- (M) $\|b_x (\sigma_x(dd') - \sigma_x(d) \sigma_x(d'))\| < \varepsilon$.

Use the Choi–Effros theorem to find a c.p.c. lift $\rho_x: \mathcal{D} \rightarrow A$ of σ_x . By upper semicontinuity of the norm function (see the discussion in 1.3 above), it follows that for any x there is some open interval I_x containing x such that ρ_x is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $\overline{I_x}$. Those intervals cover $[0, 1]$, so by compactness, we can find a finite subcover.

Now, by making the intervals smaller, and by omitting redundant elements of the cover, we obtain ψ_1, \dots, ψ_n as required. \square

We shall prove Theorem 4.6 for $X = [0, 1]$ by patching the maps ψ_1, \dots, ψ_n from Lemma 4.3 (which will be chosen with a finer auxiliary $\mathcal{G}', \varepsilon'$) into one c.p.c. map. In the following two lemmas, we will show how two c.p.c. maps $\rho, \sigma: \mathcal{D} \rightarrow A$, which are sufficiently well-behaved on adjacent intervals $[r, s]$ and $[s, t]$ respectively, can be patched together to obtain one c.p.c. map which is well-behaved on the union (where ‘well-behaved’ here means $(\mathcal{F}; \mathcal{G}, \varepsilon; \mathcal{G}', \varepsilon')$ -good for some given $\mathcal{F}, \mathcal{G}, \varepsilon, \mathcal{G}', \varepsilon'$). Since those lemmas are somewhat technical, we first give here a hand-waved outline of the proof.

In Lemma 4.4, we will show that, after perturbing ρ and σ , we can find auxiliary c.p.c. maps $\nu_\rho, \nu_\sigma: \mathcal{D} \rightarrow A$ which are well-behaved near the point s , and such that near s , we have that $\nu_\rho \approx \nu_\sigma$, that the range of ν_ρ approximately commutes with the range of ρ , and that the range of ν_σ approximately commutes with the range of σ – in fact, we obtain c.p.c. maps $\mu_\rho, \mu_\sigma: \mathcal{D} \otimes \mathcal{D} \rightarrow A$ such that $\mu_\rho \approx “\rho \otimes \nu_\rho”$ and $\mu_\sigma \approx “\sigma \otimes \nu_\sigma”$ (since the maps do not exactly commute, we cannot actually define “ $\rho \otimes \nu_\rho$ ”, “ $\sigma \otimes \nu_\sigma$ ” as c.p.c. maps).

In Lemma 4.5, we shall construct a new c.p.c. map which is well-behaved on $[r, t]$. To do so, we take a unitary path $\{u_x\}_{x \in [0, 1]} \subset \mathcal{D} \otimes \mathcal{D}$, such that $u_0 = \mathbf{1}_{\mathcal{D} \otimes \mathcal{D}}$ and $u_1(d \otimes \mathbf{1}_{\mathcal{D}})u_1^* \approx \mathbf{1}_{\mathcal{D}} \otimes d$. We then have that $\mu_\rho(u_0(d \otimes \mathbf{1}_{\mathcal{D}})u_0^*) \approx \rho(d)$, and $\mu_\rho(u_1(d \otimes \mathbf{1}_{\mathcal{D}})u_1^*) \approx \nu_\rho(d) \approx \nu_\sigma(d)$. So, in this way, we can ‘connect’ ρ to ν_ρ ($\approx \nu_\sigma$) ‘along’ the path of unitaries, and then similarly ‘connect’ ν_σ to σ . We thus obtain the desired c.p.c. map ψ , which agrees with ρ near r , with σ near t , and is roughly ν_ρ ($\approx \nu_\sigma$) near the midpoint s .

We finally note that if we were to restrict ourselves to unital C^* -algebras, the computations would become somewhat simpler. The reader who is happy to make this assumption in first reading should replace “c.p.c.” by “u.c.p.” throughout (one can then omit condition (U), and omit the “b” from condition (M)).

4.4 LEMMA: *Let \mathcal{D} be a strongly self-absorbing C^* -algebra, and let A be a separable $\mathcal{C}([0, 1])$ -algebra. Suppose we have $\varepsilon > 0$ and finite sets $\mathcal{F} \subset A$, $\mathbf{1}_{\mathcal{D}} \in \mathcal{G} \subset \mathcal{D}$ consisting of elements of norm at most one, with $\mathcal{F} = \mathcal{F}^*$, $\mathcal{G} = \mathcal{G}^*$. Suppose we have points $0 \leq r < s < t \leq 1$ and two c.p.c. maps $\rho, \sigma: \mathcal{D} \rightarrow A$ which are $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[r, s]$, $[s, t]$ respectively. Suppose furthermore that A_s is \mathcal{D} -stable.*

It follows that there are c.p.c. maps $\rho', \sigma': \mathcal{D} \rightarrow A$ which are $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[r, s]$, $[s, t]$ respectively, and c.p.c. maps $\nu_{\rho'}, \nu_{\sigma'}: \mathcal{D} \rightarrow A$, $\mu_{\rho'}, \mu_{\sigma'}: \mathcal{D} \otimes \mathcal{D} \rightarrow A$ such that $\nu_{\rho'}$ and $\nu_{\sigma'}$ are $(\mathcal{F}; \mathcal{G}, 3\varepsilon)$ -good for some interval $I \subset (r, t)$ containing s in its interior, and such that for any $b \in \mathcal{F}$ and any $d, d' \in \mathcal{G}$ we have

- (i) $_{\rho}$ $\|(b[\rho'(d), \nu_{\rho'}(d')])_I\| < 2\varepsilon$
- (i) $_{\sigma}$ $\|(b[\sigma'(d), \nu_{\sigma'}(d')])_I\| < 2\varepsilon$
- (ii) $_{\rho}$ $\|(b(\rho'(d)\nu_{\rho'}(d') - \mu_{\rho'}(d \otimes d'))_I\| < \varepsilon$
- (ii) $_{\sigma}$ $\|(b(\sigma'(d)\nu_{\sigma'}(d') - \mu_{\sigma'}(d \otimes d'))_I\| < \varepsilon$
- (iii) $\|(b(\nu_{\rho'}(d) - \nu_{\sigma'}(d)))_I\| < 2\varepsilon$.

If ρ, σ are $(\mathcal{F}; \mathcal{G}, \varepsilon; \mathcal{G}', \varepsilon')$ -good for $[r, s], [s, t]$ respectively, for some finite self-adjoint set $\mathcal{G}' \supset \mathcal{G}$ in the unit ball of \mathcal{D} and for some $0 < \varepsilon' < \varepsilon$, then one can arrange

that so are ρ', σ' , that ν_ρ and ν_σ are $(\mathcal{F}; \mathcal{G}', 3\varepsilon')$ -good for the interval I , and that conditions (i) $_\rho, (i)_\sigma, (ii)_\rho, (ii)_\sigma, (iii)$ hold with ε' instead of ε , and \mathcal{G}' instead of \mathcal{G} .

PROOF: Denote $\mathcal{G} \cdot \mathcal{G} = \{ab \mid a, b \in \mathcal{G}\}$, then $\mathcal{G} \cdot \mathcal{G} \supset \mathcal{G}$. Since \mathcal{F}, \mathcal{G} are finite, it follows that there is some $\varepsilon > \eta > 0$ such that the maps ρ, σ are also $(\mathcal{F}; \mathcal{G}, \eta)$ -good for $[r, s], [s, t]$, respectively. We use Proposition 4.1 c) to find *-homomorphisms $\kappa: A_s \rightarrow A_s$ and $\mu: \mathcal{D} \rightarrow \mathcal{M}(A_s)$ such that

$$(3) \quad [\kappa(A_s), \mu(\mathcal{D})] = 0$$

and

$$(4) \quad \|\kappa(a_s) - a_s\| < (\varepsilon - \eta)/3$$

for all $a \in \mathcal{F} \cup \rho(\mathcal{G} \cdot \mathcal{G}) \cup \sigma(\mathcal{G} \cdot \mathcal{G})$. This in particular implies that

$$(5) \quad \|\mu(d), b_s\| < \|\mu(d), \kappa(b_s)\| + 2(\varepsilon - \eta)/3 = 2(\varepsilon - \eta)/3 < \varepsilon$$

for all $b \in \mathcal{F}, d \in \mathcal{G}$. Use the Choi–Effros theorem to find c.p.c. lifts $\tilde{\rho}, \tilde{\sigma}: \mathcal{D} \rightarrow A$ for the maps $\kappa \circ \pi_s \circ \rho, \kappa \circ \pi_s \circ \sigma$, respectively. Using (4), we get

$$(U)_{\kappa \circ \sigma} \quad \|b_s \kappa(\sigma(\mathbf{1}_{\mathcal{D}})_s) - b_s\| < \eta + \frac{\varepsilon - \eta}{3} < \varepsilon$$

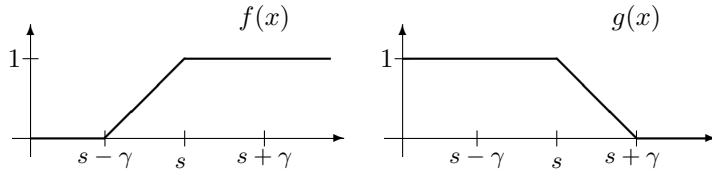
$$(C)_{\kappa \circ \sigma} \quad \|b_s \kappa(\sigma(d)_s) - \kappa(\sigma(d)_s) b_s\| \leq \|[b_s, \sigma(d)_s]\| + 2 \cdot \|\sigma(d)_s - \kappa(\sigma(d)_s)\| \\ < \eta + 2 \cdot \frac{\varepsilon - \eta}{3} < \varepsilon$$

$$(M)_{\kappa \circ \sigma} \quad \|b_s(\kappa(\sigma(dd')_s) - \kappa(\sigma(d)_s)\kappa(\sigma(d')_s))\| < \eta + 3 \cdot \frac{\varepsilon - \eta}{3} = \varepsilon$$

for all $b \in \mathcal{F}, d, d' \in \mathcal{G}$ and the same estimates hold for ρ instead of σ . Thus, by upper semicontinuity of the norm function, we have that there is some $\gamma > 0$ such that $\tilde{\rho}, \tilde{\sigma}$ are $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $I := [s - \gamma, s + \gamma]$, and such that

$$(6) \quad \|(\rho(d) - \tilde{\rho}(d))_{[s-\gamma, s+\gamma]}\|, \|(\sigma(d) - \tilde{\sigma}(d))_{[s-\gamma, s+\gamma]}\| < \frac{\varepsilon - \eta}{3}$$

for all $d \in \mathcal{G} \cdot \mathcal{G}$. We may assume that $r < s - \gamma < s + \gamma < t$. Define $f, g \in \mathcal{C}([0, 1])$ as follows.



Now, define

$$\rho'(d) := (1 - f) \cdot \rho(d) + f \cdot \tilde{\rho}(d) \text{ and } \sigma'(d) := (1 - g) \cdot \sigma(d) + g \cdot \tilde{\sigma}(d).$$

It is clear that ρ', σ' are c.p.c. We now show that ρ' is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[r, s]$ and that σ' is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[s, t]$. We establish this for ρ' – the proof for σ' is the same.

Note that $\rho'(d)_{[0, s-\gamma]} = \rho(d)_{[0, s-\gamma]}$, and thus it remains to show that ρ' is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[s - \gamma, s]$. Indeed, using (6), we obtain

$$\begin{aligned}
(\text{U})_{\rho'} & \| (b\rho'(\mathbf{1}_{\mathcal{D}}) - b)_{[s-\gamma, s]} \| \\
& < \| (b\rho(\mathbf{1}_{\mathcal{D}}) - b)_{[s-\gamma, s]} \| + \frac{\varepsilon - \eta}{3} < \eta + \frac{\varepsilon - \eta}{3} < \varepsilon \\
(\text{C})_{\rho'} & \| (b\rho'(d) - \rho'(d)b)_{[s-\gamma, s]} \| \\
& < \| (b\rho(d) - \rho(d)b)_{[s-\gamma, s]} \| + 2 \cdot \frac{\varepsilon - \eta}{3} < \eta + 2 \cdot \frac{\varepsilon - \eta}{3} < \varepsilon \\
(\text{M})_{\rho'} & \| (b(\rho'(dd') - \rho'(d)\rho'(d')))_{[s-\gamma, s]} \| \\
& < \| (b(\rho(dd') - \rho(d)\rho(d')))_{[s-\gamma, s]} \| + 3 \cdot \frac{\varepsilon - \eta}{3} < \eta + \varepsilon - \eta = \varepsilon
\end{aligned}$$

for all $b \in \mathcal{F}$, $d, d' \in \mathcal{G}$, as required.

Now, note that by (3), the range of μ commutes with $\rho'(d)_s$ and $\sigma'(d)_s$ for all $d \in \mathcal{D}$. We may thus define c.p.c. maps $\bar{\mu}_{\rho'}, \bar{\mu}_{\sigma'} : \mathcal{D} \otimes \mathcal{D} \rightarrow A_s$ by

$$\bar{\mu}_{\rho'}(d \otimes d') := \rho'(d)_s \mu(d') \quad \text{and} \quad \bar{\mu}_{\sigma'}(d \otimes d') := \sigma'(d)_s \mu(d')$$

and we define c.p.c. maps $\bar{\nu}_{\rho'}, \bar{\nu}_{\sigma'} : \mathcal{D} \rightarrow A_s$ by

$$\bar{\nu}_{\rho'}(d) := \rho'(\mathbf{1}_{\mathcal{D}})_s \mu(d') = \bar{\mu}_{\rho'}(\mathbf{1}_{\mathcal{D}} \otimes d') \quad \text{and} \quad \bar{\nu}_{\sigma'}(d) := \sigma'(\mathbf{1}_{\mathcal{D}})_s \mu(d') = \bar{\mu}_{\sigma'}(\mathbf{1}_{\mathcal{D}} \otimes d').$$

We now use the Choi–Effros theorem to choose some c.p.c. lifts $\mu_{\rho'}, \mu_{\sigma'}, \nu_{\rho'}, \nu_{\sigma'}$ into A for $\bar{\mu}_{\rho'}, \bar{\mu}_{\sigma'}, \bar{\nu}_{\rho'}, \bar{\nu}_{\sigma'}$. We need to show that $\nu_{\rho'}$ and $\nu_{\sigma'}$ are $(\mathcal{F}; \mathcal{G}, 3\varepsilon)$ -good for some neighborhood of s , and that the five conditions in the statement hold. By upper semicontinuity, it suffices to verify this at s .

We first show that $\nu_{\rho'}$ is $(\mathcal{F}; \mathcal{G}, 3\varepsilon)$ -good for some neighborhood of s ; the proof for $\nu_{\sigma'}$ is the same, so we omit it. Indeed, since ρ' is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $\{s\}$, we have

$$\begin{aligned}
(\text{U})_{\nu_{\rho'}} & \| b_s \nu_{\rho'}(\mathbf{1}_{\mathcal{D}})_s - b_s \| = \| b_s \rho'(\mathbf{1}_{\mathcal{D}})_s - b_s \| < \varepsilon \\
(\text{C})_{\nu_{\rho'}} & \| b_s \nu_{\rho'}(d)_s - \nu_{\rho'}(d)_s b_s \| = \| b_s \rho'(\mathbf{1}_{\mathcal{D}})_s \mu(d) - \underbrace{\rho'(\mathbf{1}_{\mathcal{D}})_s \mu(d)}_{=\mu(d)\rho'(\mathbf{1}_{\mathcal{D}})_s} b_s \| \\
& \leq \underbrace{\| b_s \rho'(\mathbf{1}_{\mathcal{D}})_s - b_s \|}_{< \varepsilon \text{ by (U)}_{\rho'}} + \underbrace{\| \rho'(\mathbf{1}_{\mathcal{D}})_s b_s - b_s \|}_{< \varepsilon \text{ by (U)}_{\rho'}} + \underbrace{\| b_s \mu(d) - \mu(d) b_s \|}_{< \varepsilon \text{ by (5)}} < 3\varepsilon \\
(\text{M})_{\nu_{\rho'}} & \| b_s (\nu_{\rho'}(dd')_s - \nu_{\rho'}(d)_s \nu_{\rho'}(d')_s) \| \\
& = \| b_s \rho'(\mathbf{1}_{\mathcal{D}})_s (\mu(dd') - \underbrace{\mu(d)\rho'(\mathbf{1}_{\mathcal{D}})_s \mu(d')}_{=\rho'(\mathbf{1}_{\mathcal{D}})_s \mu(d)}) \| \\
& = \| b_s \rho'(\mathbf{1}_{\mathcal{D}})_s \underbrace{(\mu(dd') - \mu(d)\mu(d'))}_{=0} \| \\
& \quad + \| (b_s - b_s \rho'(\mathbf{1}_{\mathcal{D}})_s) \rho'(\mathbf{1}_{\mathcal{D}})_s \mu(d) \mu(d') \| \\
& \leq \| b_s - b_s \rho'(\mathbf{1}_{\mathcal{D}})_s \| < \varepsilon.
\end{aligned}$$

We now verify condition (i) $_{\rho'}$:

$$\begin{aligned}
& \| (b_s [\rho'(d)_s, \nu_{\rho'}(d')_s]) \| = \| b_s [\rho'(d)_s, \rho'(\mathbf{1}_{\mathcal{D}})_s] \mu(d') \| \\
& \leq \underbrace{\| b_s (\rho'(d)_s \rho'(\mathbf{1}_{\mathcal{D}})_s - \rho'(d)_s) \|}_{< \varepsilon \text{ by (M)}_{\rho'}} + \underbrace{\| b_s (\rho'(\mathbf{1}_{\mathcal{D}})_s \rho'(d)_s - \rho'(d)_s) \|}_{< \varepsilon \text{ by (M)}_{\rho'}} < 2\varepsilon.
\end{aligned}$$

As for condition (ii) $_{\rho}$, we have indeed (again using (M) $_{\rho'}$):

$$\|(b_s(\rho'(d)_s\nu_{\rho'}(d')_s - \mu_{\rho'}(d \otimes d')_s)\| = \|(b_s(\rho'(d)_s\rho'(\mathbf{1}_{\mathcal{D}})_s - \rho'(d)_s)\mu(d')\| < \varepsilon.$$

Conditions (i) $_{\sigma}$ and (ii) $_{\sigma}$ follow in the same manner. It remains to check condition (iii). Indeed, using condition (U) for ρ' and σ' , we have

$$\begin{aligned} \|(b_s(\nu_{\rho'}(d)_s - \nu_{\sigma'}(d)_s)\| &= \|b_s(\rho'(\mathbf{1}_{\mathcal{D}})_s - \sigma'(\mathbf{1}_{\mathcal{D}})_s)\mu(d)\| \\ &\leq \|b_s\rho'(\mathbf{1}_{\mathcal{D}})_s - b_s\| + \|b_s - b_s\sigma'(\mathbf{1}_{\mathcal{D}})_s\| < 2\varepsilon. \end{aligned}$$

The last part of the lemma follows immediately from the construction, provided that we choose $\gamma > 0$ such that ρ and σ are $(\mathcal{F}; \mathcal{G}', \varepsilon')$ -good for $[s - \gamma, s + \gamma]$. \square

4.5 LEMMA: *Let \mathcal{D} be a K_1 -injective strongly self-absorbing C^* -algebra. Let A be a separable $\mathcal{C}([0, 1])$ -algebra. Suppose we are given $\varepsilon > 0$ and finite subsets $\mathcal{F} \subset A$ and $\mathbf{1}_{\mathcal{D}} \in \mathcal{G} \subset \mathcal{D}$, with $\mathcal{F} = \mathcal{F}^*$, $\mathcal{G} = \mathcal{G}^*$.*

There exist $0 < \varepsilon' < \varepsilon$ and a finite set $\mathcal{G} \subset \mathcal{G}' \subset \mathcal{D}$, such that if we have two c.p.c. maps $\rho, \sigma: \mathcal{D} \rightarrow A$, and points $0 \leq r < s < t \leq 1$ such that ρ is $(\mathcal{F}; \mathcal{G}, \varepsilon; \mathcal{G}', \varepsilon')$ -good for $[r, s]$ and σ is $(\mathcal{F}; \mathcal{G}, \varepsilon; \mathcal{G}', \varepsilon')$ -good for $[s, t]$, then there is a c.p.c. map $\psi: \mathcal{D} \rightarrow A$ which is $(\mathcal{F}; \mathcal{G}, \varepsilon; \mathcal{G}', \varepsilon')$ -good for $[r, t]$.

PROOF: We may assume without loss of generality that the elements of \mathcal{F} and \mathcal{G} have norm at most one. Since \mathcal{D} is K_1 -injective, [34, Proposition 1.13] guarantees that we may choose a unitary $u \in \mathcal{C}([0, 1], \mathcal{D} \otimes \mathcal{D})$ such that

$$(7) \quad u_0 = \mathbf{1}_{\mathcal{D} \otimes \mathcal{D}}$$

and

$$(8) \quad \|u_1(d \otimes \mathbf{1}_{\mathcal{D}})u_1^* - \mathbf{1}_{\mathcal{D}} \otimes d\| < \frac{\varepsilon}{4}$$

for all $d \in \mathcal{G} \cdot \mathcal{G} = \{d_1 d_2 \mid d_1, d_2 \in \mathcal{G}\}$. Since the set $\{u_x\}_{x \in [0, 1]}$ is compact, we may now fix some $m \in \mathbb{N}$ and v_{ji}, w_{ji} , $i, j = 1, \dots, m$ in the unit ball of \mathcal{D} , such that for any $x \in [0, 1]$ there is an i with

$$\left\| u_x - \sum_{j=1}^m v_{ji} \otimes w_{ji} \right\| < \frac{\varepsilon}{9}.$$

Denote $y_i := \sum_{j=1}^m v_{ji} \otimes w_{ji}$. We may assume that $\|y_i\| \leq 1$ for all i .

Now, set

$$\mathcal{G}' := \{d_1 d_2 d_3 d_4 d_5 d_6 \mid d_1, \dots, d_6 \in \mathcal{G} \cup \{v_{ji}, v_{ji}^*, w_{ji}, w_{ji}^* \mid i, j = 1, \dots, m\}\}$$

and

$$\varepsilon' := \frac{\varepsilon}{144m^4}.$$

Suppose we are given ρ, σ as in the statement. We wish to construct the desired c.p.c. map ψ . By replacing ρ, σ by ρ', σ' as in the Lemma 4.4 if needed, we have c.p.c. maps $\nu_{\rho}, \nu_{\sigma}: \mathcal{D} \rightarrow A$ and $\mu_{\rho}, \mu_{\sigma}: \mathcal{D} \otimes \mathcal{D} \rightarrow A$ such that the conclusions of the lemma hold for some interval $I \subset (r, t)$ containing s in its interior. By upper semicontinuity of the norm function, there is some $\delta > 0$ such that both ρ and σ are $(\mathcal{F}; \mathcal{G}', \varepsilon')$ -good for $[s - 3\delta, s + 3\delta] \subset I$.

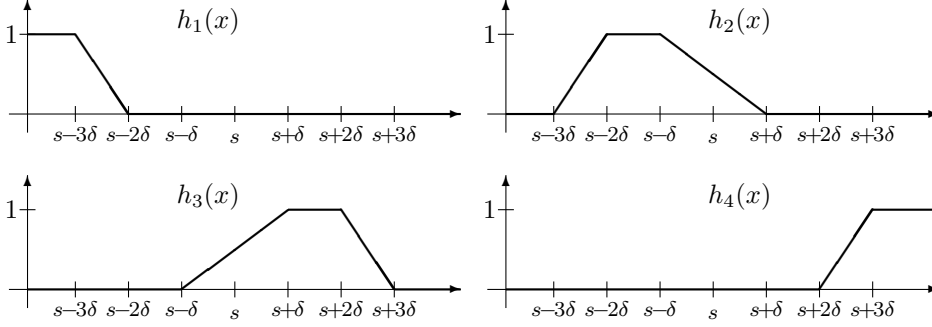
Define c.p.c. maps

$$\varphi_{\rho}, \varphi_{\sigma}: \mathcal{C}([0, 1]) \otimes \mathcal{D} \otimes \mathcal{D} \rightarrow A$$

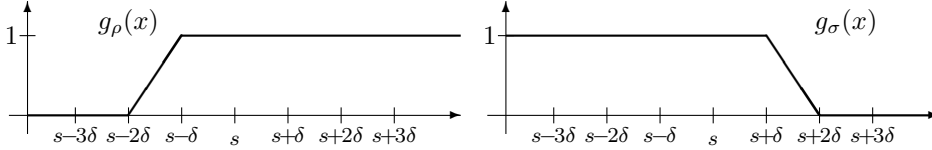
by

$$\varphi_\rho(f \otimes d \otimes d') := f \cdot \mu_\rho(d \otimes d'), \quad \varphi_\sigma(f \otimes d \otimes d') := f \cdot \mu_\sigma(d \otimes d').$$

Define continuous functions $h_1, h_2, h_3, h_4: [0, 1] \rightarrow [0, 1]$ which sum up to 1, by:



Define $g_\rho, g_\sigma: [0, 1] \rightarrow [0, 1]$ by



Define unitaries $u_\rho, u_\sigma \in \mathcal{C}([0, 1]) \otimes \mathcal{D} \otimes \mathcal{D} \cong \mathcal{C}([0, 1], \mathcal{D} \otimes \mathcal{D})$ by

$$u_{\rho x} := u_{g_\rho(x)}, \quad u_{\sigma x} := u_{g_\sigma(x)}.$$

Define c.p.c. maps $\zeta_\rho, \zeta_\sigma: \mathcal{D} \rightarrow A$ by

$$\zeta_\rho(d) := \varphi_\rho(u_\rho(\mathbf{1}_{\mathcal{C}([0,1])} \otimes d \otimes \mathbf{1}_{\mathcal{D}})u_\rho^*), \quad \zeta_\sigma(d) := \varphi_\sigma(u_\sigma(\mathbf{1}_{\mathcal{C}([0,1])} \otimes d \otimes \mathbf{1}_{\mathcal{D}})u_\sigma^*).$$

Finally, we define $\psi: \mathcal{D} \rightarrow A$ by

$$\psi(d) := h_1 \cdot \rho(d) + h_2 \cdot \zeta_\rho(d) + h_3 \cdot \zeta_\sigma(d) + h_4 \cdot \sigma(d).$$

ψ is clearly a c.p.c. map. Note that $\psi(d)_{[0, s-3\delta]} = \rho(d)_{[0, s-3\delta]}$ and $\psi(d)_{[s+3\delta, 1]} = \sigma(d)_{[s+3\delta, 1]}$. In particular, it follows that ψ is $(\mathcal{F}; \mathcal{G}', \varepsilon')$ -good for some neighborhood of the endpoints of the interval $[r, t]$, and is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[r, s-3\delta] \cup [s+3\delta, t]$.

It remains to show that ψ is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[s-3\delta, s+3\delta]$. To verify condition (U) in Definition 4.2, note that

$$\psi(\mathbf{1}_{\mathcal{D}}) = h_1 \cdot \rho(\mathbf{1}_{\mathcal{D}}) + h_2 \cdot \mu_\rho(\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}) + h_3 \cdot \mu_\sigma(\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}) + h_4 \cdot \sigma(\mathbf{1}_{\mathcal{D}}).$$

Thus, for any $b \in \mathcal{F}$ and any $x \in [s - 3\delta, s + 3\delta]$ we have

$$\begin{aligned}
& \|(b\psi(\mathbf{1}_{\mathcal{D}}) - b)_x\| \\
& \leq h_1(x) \underbrace{\|(b\rho(\mathbf{1}_{\mathcal{D}}) - b)_x\|}_{< \varepsilon'} \\
& \quad + h_2(x) \cdot \underbrace{\|(\underbrace{(b(\mu_\rho(\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}) - \rho(\mathbf{1}_{\mathcal{D}})\nu_\rho(\mathbf{1}_{\mathcal{D}})))_x}_{< \varepsilon' \text{ by 4.4 (ii)}_\rho} - b)_x\|}_{< \varepsilon'} + \|(b\rho(\mathbf{1}_{\mathcal{D}})\nu_\rho(\mathbf{1}_{\mathcal{D}}) - b)_x\| \\
& \quad + h_3(x) \cdot \underbrace{\|(\underbrace{(b(\mu_\sigma(\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}) - \sigma(\mathbf{1}_{\mathcal{D}})\nu_\sigma(\mathbf{1}_{\mathcal{D}})))_x}_{< \varepsilon' \text{ by 4.4 (ii)}_\sigma} - b)_x\|}_{< \varepsilon'} + \|(b\sigma(\mathbf{1}_{\mathcal{D}})\nu_\sigma(\mathbf{1}_{\mathcal{D}}) - b)_x\| \\
& \quad + h_4(x) \underbrace{\|(b\sigma(\mathbf{1}_{\mathcal{D}}) - b)_x\|}_{< \varepsilon'} \\
& < \underbrace{(h_1(x) + h_2(x) + h_3(x) + h_4(x))}_{=1} \varepsilon' \\
& \quad + h_2(x) \cdot \|(b\rho(\mathbf{1}_{\mathcal{D}})\nu_\rho(\mathbf{1}_{\mathcal{D}}) - b)_x\| + h_3(x) \cdot \|(b\sigma(\mathbf{1}_{\mathcal{D}})\nu_\sigma(\mathbf{1}_{\mathcal{D}}) - b)_x\| \\
& \leq h_2(x) \cdot \underbrace{\|(b\rho(\mathbf{1}_{\mathcal{D}}) - b)_x\|}_{< \varepsilon'} \|\nu_\rho(\mathbf{1}_{\mathcal{D}})_x\| + \underbrace{\|(b\nu_\rho(\mathbf{1}_{\mathcal{D}}) - b)_x\|}_{< 3\varepsilon'} \\
& \quad + h_3(x) \cdot \underbrace{\|(b\sigma(\mathbf{1}_{\mathcal{D}}) - b)_x\|}_{< \varepsilon'} \|\nu_\sigma(\mathbf{1}_{\mathcal{D}})_x\| + \underbrace{\|(b\nu_\sigma(\mathbf{1}_{\mathcal{D}}) - b)_x\|}_{< 3\varepsilon'} + \varepsilon' \\
& < (h_2(x) + h_3(x)) \cdot 4\varepsilon' + \varepsilon' \leq 5\varepsilon' < \varepsilon.
\end{aligned}$$

For condition (C), let $b \in \mathcal{F}$, $d \in \mathcal{G}$, $x \in [s - 3\delta, s + 3\delta]$. We wish to show that $\|b_x\psi(d)_x - \psi(d)_x b_x\| < \varepsilon$. By the definition of ψ , and the fact that ρ, σ are $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $\{x\}$, it will suffice to show that

$$\|b_x\zeta_\rho(d)_x - \zeta_\rho(d)_x b_x\| < \varepsilon \text{ and } \|b_x\zeta_\sigma(d)_x - \zeta_\sigma(d)_x b_x\| < \varepsilon$$

We will show it for ζ_ρ – the proof for ζ_σ is similar. If $x \in [s - 3\delta, s - 2\delta]$ then $\zeta_\rho(d)_x = \mu_\rho(d)_x$, and hence $\|b_x(\zeta_\rho(d)_x - \rho(d)_x\nu_\rho(d)_x)\| < \varepsilon'$. Thus, we have

$$\begin{aligned}
& \|b_x\zeta_\rho(d)_x - \zeta_\rho(d)_x b_x\| < \|[b_x, \rho(d)_x\nu_\rho(d)_x]\| + 2\varepsilon' \\
& \leq \underbrace{\|[b_x, \rho(d)_x]\|}_{< \varepsilon'} + \underbrace{\|[b_x, \nu_\rho(d)_x]\|}_{< 3\varepsilon'} + 2\varepsilon' < 6\varepsilon' < \varepsilon.
\end{aligned}$$

If $x \in [s - \delta, s + 3\delta]$, we have (by (8))

$$\|\zeta_\rho(d)_x - \mu_\rho(\mathbf{1}_{\mathcal{D}} \otimes d)_x\| < \frac{\varepsilon}{4}$$

and by 4.4 (ii) $_\rho$,

$$\|b_x(\mu_\rho(\mathbf{1}_{\mathcal{D}} \otimes d)_x - \rho(\mathbf{1}_{\mathcal{D}})_x\nu_\rho(d)_x)\|, \|(\mu_\rho(\mathbf{1}_{\mathcal{D}} \otimes d)_x - \rho(\mathbf{1}_{\mathcal{D}})_x\nu_\rho(d)_x)b_x\| < \varepsilon'.$$

So, as in the previous consideration, we have

$$\|b_x\zeta_\rho(d)_x - \zeta_\rho(d)_x b_x\| < \|[b_x, \rho(\mathbf{1}_{\mathcal{D}})_x\nu_\rho(d)_x]\| + \frac{2\varepsilon}{4} + 2\varepsilon' < \frac{2\varepsilon}{4} + 6\varepsilon' \leq \varepsilon.$$

Finally, if $x \in [s - 2\delta, s - \delta]$, find i such that

$$(9) \quad \|u_{\rho x} - y_i\| < \frac{\varepsilon}{9}.$$

We then have

$$(10) \quad \|\zeta_\rho(d)_x - \mu_\rho(y_i(d \otimes \mathbf{1}_{\mathcal{D}})y_i^*)_x\| < \frac{2\varepsilon}{9}.$$

Note that

$$(11) \quad \mu_\rho(y_i(d \otimes \mathbf{1}_{\mathcal{D}})y_i^*) = \sum_{j,k=1}^m \mu_\rho(v_{ji}dv_{ki}^* \otimes w_{ji}w_{ki}^*)$$

and for any $1 \leq j, k \leq m$, we have (by 4.4 (ii) $_\rho$)

$$(12) \quad \|b_x(\mu_\rho(v_{ji}dv_{ki}^* \otimes w_{ji}w_{ki}^*) - \rho(v_{ji}dv_{ki}^*)\nu_\rho(w_{ji}w_{ki}^*))_x\| < \varepsilon'$$

and we get a similar estimate by placing b_x on the right rather than the left. Thus,

$$\begin{aligned} & \|b_x \zeta_\rho(d)_x - \zeta_\rho(d)_x b_x\| \\ & < \| [b_x, \mu_\rho(y_i(d \otimes \mathbf{1}_{\mathcal{D}})y_i^*)]_x \| + \frac{4\varepsilon}{9} \\ & < \sum_{j,k=1}^m \| [b_x, \rho(v_{ji}dv_{ki}^*)_x \nu_\rho(w_{ji}w_{ki}^*)_x] \| + \frac{4\varepsilon}{9} + 2m^2\varepsilon' \\ & \leq \sum_{j,k=1}^m \left(\underbrace{\| [b_x, \rho(v_{ji}dv_{ki}^*)_x] \|}_{< \varepsilon'} + \underbrace{\| [b_x, \nu_\rho(w_{ji}w_{ki}^*)_x] \|}_{< 3\varepsilon'} \right) + \frac{4\varepsilon}{9} + 2m^2\varepsilon' \\ & < 6m^2\varepsilon' + \frac{4\varepsilon}{9} \leq \varepsilon. \end{aligned}$$

For condition (M), let $b \in \mathcal{F}$, $d, d' \in \mathcal{G}$, $x \in [s - 3\delta, s + 3\delta]$. We must show that

$$\|(b(\psi(dd') - \psi(d)\psi(d')))_x\| < \varepsilon.$$

We shall just show it for $x \in [s - 3\delta, s]$ – the case $x \in [s, s + 3\delta]$ is obtained similarly, where the roles of ρ and σ are reversed.

If $x \in [s - 3\delta, s - 2\delta]$, We have that

$$\psi(c)_x = h_1(x)\rho(c)_x + h_2(x)\mu_\rho(c \otimes \mathbf{1}_{\mathcal{D}})_x.$$

For any $c \in \mathcal{G}'$, $b \in \mathcal{F}$, we have (4.4 (ii) $_\rho$)

$$\|b_x(\mu_\rho(c \otimes \mathbf{1}_{\mathcal{D}})_x - \rho(c)_x \nu_\rho(\mathbf{1}_{\mathcal{D}})_x)\| < \varepsilon'$$

and furthermore

$$\begin{aligned} & \|b_x(\rho(c)_x \nu_\rho(\mathbf{1}_{\mathcal{D}})_x - \rho(c)_x)\| \\ & \leq \underbrace{\|b_x \nu_\rho(\mathbf{1}_{\mathcal{D}})_x - b_x\|}_{< 3\varepsilon'} \|\rho(c)_x\| + \underbrace{\|b_x[\rho(c)_x, \nu_\rho(\mathbf{1}_{\mathcal{D}})_x]\|}_{< 2\varepsilon' \text{ by 4.4 (i)}_\rho} < 5\varepsilon'. \end{aligned}$$

Thus, we see that

$$(13) \quad \|b_x(\mu_\rho(c \otimes \mathbf{1}_{\mathcal{D}})_x - \rho(c)_x)\|, \|b_x(\psi(c)_x - \rho(c)_x)\| < 6\varepsilon'.$$

So,

$$\begin{aligned} & \|(b(\psi(dd') - \psi(d)\psi(d')))_x\| \\ & < \|(b(\rho(dd') - \rho(d)\mu_\rho(d' \otimes \mathbf{1}_{\mathcal{D}})))_x\| + 12\varepsilon' \\ & \leq \underbrace{\|(b(\rho(dd') - \rho(d)\rho(d')))_x\|}_{< \varepsilon'} + \|b_x \rho(d)_x(\rho(d') - \mu_\rho(d' \otimes \mathbf{1}_{\mathcal{D}}))\| + 12\varepsilon' \\ & < \underbrace{\|\rho(d)_x b_x(\rho(d') - \mu_\rho(d' \otimes \mathbf{1}_{\mathcal{D}}))\|}_{< 6\varepsilon' \text{ by (13)}} + \underbrace{\|[b_x, \rho(d)_x]\|}_{< \varepsilon'} + 13\varepsilon' \\ & < 20\varepsilon' < \varepsilon. \end{aligned}$$

If $x \in [s - 2\delta, s - \delta]$, we have $\psi(c)_x = \zeta_\rho(c)_x = \mu_\rho(u_{\rho x}(c \otimes \mathbf{1}_D)u_{\rho x}^*)$. Repeating the considerations from equations (9)–(12) above, we find an i such that

$$(14) \quad \|u_{\rho x} - y_i\| < \frac{\varepsilon}{9}$$

and then we have

$$\begin{aligned} & \| (b(\psi(dd') - \psi(d)\psi(d')))_{x} \| \\ & < \left\| b_x \left(\zeta_\rho(dd') - \sum_{j,k,\ell,n=1}^m \mu_\rho(v_{ji}d'v_{ki}^* \otimes w_{ji}w_{ki}^*) \mu_\rho(v_{\ell i}d'v_{ni}^* \otimes w_{\ell i}w_{ni}^*) \right) \right\|_x + \frac{4\varepsilon}{9}. \end{aligned}$$

Now, note that if $c_1, c_2, c_3, c_4 \in \mathcal{G}'$ (which we will apply as $c_1 = v_{ji}d'v_{ki}^*$, $c_2 = w_{ji}w_{ki}^*$, $c_3 = v_{\ell i}d'v_{ni}^*$, $c_4 = w_{\ell i}w_{ni}^*$), we have (by 4.4 (ii) $_\rho$)

$$\|b_x(\mu_\rho(c_1 \otimes c_2)_x - \rho(c_1)_x \nu_\rho(c_2)_x)\| < \varepsilon'$$

and therefore

$$\begin{aligned} (15) \quad & \|b_x(\mu_\rho(c_1 \otimes c_2)_x \mu_\rho(c_3 \otimes c_4)_x - \rho(c_1)_x \nu_\rho(c_2)_x \rho(c_3)_x \nu_\rho(c_4)_x)\| \\ & < \|b_x \rho(c_1)_x \nu_\rho(c_2)_x (\mu_\rho(c_3 \otimes c_4)_x - \rho(c_3)_x \nu_\rho(c_4)_x)\| + \varepsilon' \\ & < \|b_x(\mu_\rho(c_3 \otimes c_4)_x - \rho(c_3)_x \nu_\rho(c_4)_x)\| + \|[b_x, \rho(c_1)_x \nu_\rho(c_2)_x]\| + \varepsilon' \\ & < \underbrace{\|[b_x, \rho(c_1)_x]\|}_{< \varepsilon'} + \underbrace{\|[b_x, \nu_\rho(c_2)_x]\|}_{< 3\varepsilon'} + 2\varepsilon' < 6\varepsilon'. \end{aligned}$$

Thus, we have that

$$\begin{aligned} (16) \quad & \| (b(\psi(dd') - \psi(d)\psi(d')))_{x} \| \\ & < \left\| b_x \left(\zeta_\rho(dd') - \sum_{j,k,\ell,n=1}^m \rho(v_{ji}d'v_{ki}^*) \nu_\rho(w_{ji}w_{ki}^*) \rho(v_{\ell i}d'v_{ni}^*) \nu_\rho(w_{\ell i}w_{ni}^*) \right) \right\|_x \\ & + \frac{4\varepsilon}{9} + 6m^4\varepsilon'. \end{aligned}$$

Now, note that

$$\begin{aligned} & \|b_x(\rho(c_1)\nu_\rho(c_2)\rho(c_3)\nu_\rho(c_4) - \rho(c_1)\rho(c_3)\nu_\rho(c_2)\nu_\rho(c_4))_x\| \\ & \leq \underbrace{\|[b_x, \rho(c_1)_x]\|}_{< \varepsilon'} + \underbrace{\|[b_x[\nu_\rho(c_2)_x, \rho(c_3)_x]]\|}_{< 2\varepsilon'} < 3\varepsilon' \end{aligned}$$

and

$$\begin{aligned} & \|b_x(\rho(c_1)\rho(c_3)\nu_\rho(c_2)\nu_\rho(c_4) - \rho(c_1c_3)\nu_\rho(c_2c_4))_x\| \\ & \leq \|b_x \rho(c_1)_x \rho(c_3)_x (\nu_\rho(c_2)\nu_\rho(c_4) - \nu_\rho(c_2c_4))_x\| + \underbrace{\|b_x(\rho(c_1)\rho(c_3) - \rho(c_1c_3))_x\|}_{< \varepsilon'} \\ & < \|[b_x, \rho(c_1)_x \rho(c_3)_x]\| + \underbrace{\|b_x(\nu_\rho(c_4) - \nu_\rho(c_2c_4))_x\|}_{< 3\varepsilon'} + \varepsilon' \\ & < \|[b_x, \rho(c_1)_x]\| + \|[b_x, \rho(c_3)_x]\| + 4\varepsilon' < 6\varepsilon' \end{aligned}$$

and so

$$(17) \quad \|b_x(\rho(c_1)\nu_\rho(c_2)\rho(c_3)\nu_\rho(c_4) - \rho(c_1c_3)\nu_\rho(c_2c_4))_x\| < 9\varepsilon'.$$

Thus,

$$\begin{aligned}
& \| (b(\psi(dd') - \psi(d)\psi(d')))_x \| \\
& \stackrel{(16,17)}{<} \left\| b_x \left(\zeta_\rho(dd') - \sum_{j,k,\ell,n=1}^m \rho(v_{ji}dv_{ki}^*v_{\ell i}d'v_{ni}^*)\nu_\rho(w_{ji}w_{ki}^*w_{\ell i}w_{ni}^*) \right)_x \right\| \\
& \quad + \frac{4\varepsilon}{9} + 15m^4\varepsilon' \\
& \stackrel{4.4 \text{ (ii)}_\rho}{<} \left\| b_x \left(\zeta_\rho(dd') - \sum_{j,k,\ell,n=1}^m \mu_\rho(v_{ji}dv_{ki}^*v_{\ell i}d'v_{ni}^* \otimes w_{ji}w_{ki}^*w_{\ell i}w_{ni}^*) \right)_x \right\| \\
& \quad + \frac{4\varepsilon}{9} + 16m^4\varepsilon' \\
& = \| b_x(\zeta_\rho(dd') - \mu_\rho(y_i(d \otimes \mathbf{1}_D)y_i^*y_i(d' \otimes \mathbf{1}_D)y_i^*))_x \| + \frac{4\varepsilon}{9} + 16m^4\varepsilon' \\
& \stackrel{(14)}{<} \| b_x(\zeta_\rho(dd') - \underbrace{\mu_\rho(u_{\rho x}(d \otimes \mathbf{1}_D)u_{\rho x}^*u_{\rho x}(d' \otimes \mathbf{1}_D)u_{\rho x}^*)}_{=\zeta_\rho(dd')})_x \| \\
& \quad + \frac{8\varepsilon}{9} + 16m^4\varepsilon' \\
& = \frac{8\varepsilon}{9} + 16m^4\varepsilon' = \varepsilon.
\end{aligned}$$

Finally, if $x \in [s - \delta, s]$, we have that

$$\begin{aligned}
\psi(c)_x &= h_2(x)\zeta_\rho(c)_x + h_3(x)\zeta_\sigma(c)_x \\
&= h_2(x)\mu_\rho(u_1(c \otimes \mathbf{1}_D)u_1^*)_x + h_3(x)\mu_\sigma(u_1(c \otimes \mathbf{1}_D)u_1^*)_x.
\end{aligned}$$

Note that if $c \in \mathcal{G} \cdot \mathcal{G} \subseteq \mathcal{G}'$, we have (by (8))

$$\| \mu_\rho(u_1(c \otimes \mathbf{1}_D)u_1^*) - \mu_\rho(\mathbf{1}_D \otimes c) \| < \frac{\varepsilon}{4}$$

and if $b \in \mathcal{F}$, then we have (using 4.4 (ii) $_\rho$ and condition (U) for ρ)

$$\begin{aligned}
& \| b_x(\mu_\rho(\mathbf{1}_D \otimes c)_x - \nu_\rho(c)_x) \| \\
& \leq \| b_x(\mu_\rho(\mathbf{1}_D \otimes c)_x - \rho(\mathbf{1}_D)_x\nu_\rho(c)_x) \| + \| (b_x\rho(\mathbf{1}_D)_x - b_x)\nu_\rho(c)_x \| \\
& < 2\varepsilon'.
\end{aligned}$$

Thus,

$$\| b_x(\zeta_\rho(c)_x - \nu_\rho(c)_x) \| < \frac{\varepsilon}{4} + 2\varepsilon'.$$

Similarly,

$$\| b_x(\zeta_\sigma(c)_x - \nu_\sigma(c)_x) \| < \frac{\varepsilon}{4} + 2\varepsilon'.$$

But $\| b_x(\nu_\rho(c)_x - \nu_\sigma(c)_x) \| < 2\varepsilon'$ (by 4.4 (iii)), and thus we have

$$\| b_x(\zeta_\sigma(c)_x - \nu_\rho(c)_x) \| < \frac{\varepsilon}{4} + 4\varepsilon'.$$

Therefore,

$$\| b_x(\psi(c)_x - \nu_\rho(c)_x) \| < \frac{\varepsilon}{4} + 4\varepsilon'.$$

So, for $d, d' \in \mathcal{G}$, we have (since ν_ρ is $(\mathcal{F}; \mathcal{G}', 3\varepsilon')$ -good for $\{x\}$)

$$\begin{aligned}
 & \|b_x(\psi(dd') - \psi(d)\psi(d'))_x\| \\
 & < \|b_x(\nu_\rho(dd') - \nu_\rho(d)\psi(d'))_x\| + \frac{2\varepsilon}{4} + 8\varepsilon' \\
 & \leq \underbrace{\|b_x(\nu_\rho(dd') - \nu_\rho(d)\nu_\rho(d'))_x\|}_{< 3\varepsilon'} + \underbrace{\|[b_x, \nu_\rho(d)]_x\|}_{< 3\varepsilon'} \\
 & \quad + \|\nu_\rho(d)\| \underbrace{\|b_x(\psi(d')_x - \nu_\rho(d')_x)\|}_{< \varepsilon/4 + 4\varepsilon'} + \frac{2\varepsilon}{4} + 8\varepsilon' \\
 & < \frac{3\varepsilon}{4} + 18\varepsilon' < \varepsilon.
 \end{aligned}$$

□

4.6 THEOREM: *Let \mathcal{D} be a K_1 -injective strongly self-absorbing C^* -algebra. Let X be a locally compact metrizable space of finite covering dimension. Let A be a separable $C_0(X)$ -algebra. It follows that A is \mathcal{D} -stable if and only if A_x is \mathcal{D} -stable for each $x \in X$.*

PROOF: If A is \mathcal{D} -stable, then so are all fibres A_x , since \mathcal{D} -stability passes to quotients by [34, Corollary 3.3]. We prove the converse. By 1.9, it suffices to restrict to compact X . By [16, Theorem V.3] we may assume that X is a subset of $[0, 1]^N$ for some $N \in \mathbb{N}$. By 1.10, we can furthermore assume that $X = [0, 1]^N$ (the fibres are either the original fibres, or 0, all of which are \mathcal{D} -stable, so the hypothesis holds).

We can now proceed by induction. For $N = 1$, we use Proposition 4.1 b). Given finite sets $\mathcal{F} \subset A$ and $\mathbf{1}_{\mathcal{D}} \in \mathcal{G} \subset \mathcal{D}$, and $\varepsilon > 0$, we would like to find a c.p.c. map $\psi: \mathcal{D} \rightarrow A$ which is $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[0, 1]$. We may assume without loss of generality that \mathcal{F}, \mathcal{G} are self-adjoint and consist of elements of norm at most one.

Select $\mathcal{G}' \supset \mathcal{G}$, $\varepsilon > \varepsilon' > 0$ as in Lemma 4.5. Use Lemma 4.3 to find some natural number n , points $0 = t_0 < t_1 < \dots < t_n = 1$ and c.p.c. maps $\psi_k: \mathcal{D} \rightarrow A$ such that ψ_k is $(\mathcal{F}; \mathcal{G}', \varepsilon')$ -good for $[t_{k-1}, t_k]$. Note that any map which is $(\mathcal{F}; \mathcal{G}', \varepsilon')$ -good for some interval is trivially also $(\mathcal{F}; \mathcal{G}, \varepsilon; \mathcal{G}', \varepsilon')$ -good for the same interval. Now, by a repeated application ($n-1$ times) of Lemma 4.5, we can find a c.p.c. map $\psi: \mathcal{D} \rightarrow A$ which is $(\mathcal{F}; \mathcal{G}, \varepsilon; \mathcal{G}', \varepsilon')$ -good for $[0, 1]$. Such a map is in particular $(\mathcal{F}; \mathcal{G}, \varepsilon)$ -good for $[0, 1]$, i.e. satisfies the required conditions.

For the induction step, suppose now that $N > 1$, and that the statement is true for $X = [0, 1]^k$ for all $k < N$. We may clearly regard A as a $\mathcal{C}([0, 1])$ -algebra with fibres $A_{\{t\} \times [0, 1]^{N-1}}$ for $t \in [0, 1]$. By the base case of the induction, if $A_{\{t\} \times [0, 1]^{N-1}}$ is \mathcal{D} -stable for all $t \in [0, 1]$, then so is A . But, for each $t \in [0, 1]$, and each $x \in \{t\} \times [0, 1]^{N-1}$, $(A_{\{t\} \times [0, 1]^{N-1}})_x = A_x$ is \mathcal{D} -stable by assumption. Upon identifying $\{t\} \times [0, 1]^{N-1}$ with $[0, 1]^{N-1}$, it follows from the induction hypothesis that $A_{\{t\} \times [0, 1]^{N-1}}$ also absorbs \mathcal{D} . □

The two examples below show that one cannot drop the finite dimensionality condition from the preceding result.

4.7 EXAMPLE: We give here an example of a $\mathcal{C}(X)$ -algebra A , where $X = \prod_{n=1}^{\infty} S^2$ is infinite dimensional, whose fibres A_x are isomorphic to the CAR-algebra (the UHF-algebra of type 2^{∞}) for all $x \in X$, and such that A does not absorb this UHF-algebra. Actually, one cannot embed M_2 unittally into A .

The K_0 -group of $\mathcal{C}(S^2)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and is generated (as a \mathbb{Z} -module) by the K_0 -classes of two 1-dimensional projections e and p , where e is trivial (= constant) and where p is the ‘‘Bott projection’’. The latter projection belongs to $M_2(\mathcal{C}(S^2))$. Hence we can choose e and p to be mutually orthogonal projections in $M_3(\mathcal{C}(S^2))$. Put

$$B = (e + p)M_3(\mathcal{C}(S^2))(e + p), \quad A = \bigotimes_{n=1}^{\infty} B.$$

Note that B is a $\mathcal{C}(S^2)$ -algebra with fibres $B_x = M_2$ (the projections e and p are not equivalent in B , but e_x is equivalent to p_x for all $x \in S^2$). It follows from Lemma 1.8 that A is a $\mathcal{C}(X)$ -algebra with fibres A_x isomorphic to the CAR-algebra $\bigotimes_{n=1}^{\infty} M_2$ for all $x \in X$.

We proceed to show that one cannot embed M_2 unittally into A . We do so by showing that $[1_A] \in K_0(A)$ is not divisible by 2. By continuity of K_0 it suffices to show that the class in K_0 of the unit, call it q_m , of $\bigotimes_{n=1}^m B$ is not divisible by 2. Now,

$$\begin{aligned} q_m &= (e + p) \otimes (e + p) \otimes (e + p) \otimes \cdots \otimes (e + p) \\ &= \sum_{I \subseteq \{1, 2, \dots, m\}} p_I, \end{aligned}$$

where

$$p_I = r_1 \otimes r_2 \otimes \cdots \otimes r_m, \quad r_j = \begin{cases} p, & j \in I \\ e, & j \notin I \end{cases}.$$

We already noted that $[e]$ and $[p]$ form a basis for $K_0(\mathcal{C}(S^2))$ as a \mathbb{Z} -module. It follows from the Künneth theorem (and by induction) that the 2^m elements $[p_I]$, $I \subseteq \{1, 2, \dots, m\}$, form a basis for $K_0(\mathcal{C}((S^2)^m))$ (again as a \mathbb{Z} -module). This shows that $[q_m] = \sum_I [p_I]$ is not divisible by 2 in $K_0(\mathcal{C}((S^2)^m))$.

This example can easily be amended to give, for each UHF-algebra \mathcal{D} of infinite type, a $\mathcal{C}(X)$ -algebra A (with X as above) with fibres A_x isomorphic to \mathcal{D} for all $x \in X$ and such that A does not absorb \mathcal{D} tensorially. Indeed, one can construct A such that one cannot embed any non-trivial matrix algebra unittally into A .

Moreover, by tensoring A with \mathcal{O}_{∞} , we obtain an example with each fibre isomorphic to $\mathcal{O}_{\infty} \otimes \mathcal{D}$, where \mathcal{D} again is a UHF algebra of infinite type, but such that A does not absorb $\mathcal{O}_{\infty} \otimes \mathcal{D}$.

In [9], Dadarlat has produced a simple variation of the preceding example, which then yields a $\mathcal{C}(X)$ -algebra A with fibres all isomorphic to \mathcal{O}_2 , but such that A itself is not \mathcal{O}_2 -stable.

4.8 EXAMPLE: We give here an example of a $\mathcal{C}(X)$ -algebra A , where, as in the previous example, $X = \prod_{n=1}^{\infty} S^2$ and the fibres A_x are all isomorphic to the CAR-algebra (the UHF-algebra of type 2^{∞})—in particular, the fibres absorb the Jiang–Su algebra \mathcal{Z} (see [18])—but such that A does not absorb \mathcal{Z} . In fact, we show that

the semigroup $V(A)$ of Murray-von Neumann equivalence classes of projections in matrix algebras over A is not almost unperforated, i.e., there are elements $x, y \in V(A)$ and a natural number n such that $(n+1)x \leq ny$ but $x \not\leq y$. It then follows from [33, Corollary 4.8] that A cannot be \mathcal{Z} -absorbing.

Here, as in Example 4.7 above, we also get a $\mathcal{C}(X)$ -algebra with fibres isomorphic to the CAR-algebra, but which itself does not absorb the CAR-algebra. (Any C^* -algebra that tensorially absorbs a UHF-algebra will also absorb the Jiang–Su algebra, because any UHF-algebra absorbs the Jiang–Su algebra, cf. [18].) All the same, we included Example 4.7, as it is technically easier than the present example.

As in Example 4.7, let $p \in M_2(\mathcal{C}(S^2))$ be the one-dimensional ‘‘Bott projection’’. For each natural number m identify the two C^* -algebras $\bigotimes_{n=1}^m M_2(\mathcal{C}(S^2))$ and $M_{2^m}(\mathcal{C}((S^2)^m))$, and find in $M_{2^{m+1}}(\mathcal{C}((S^2)^m))$ mutually orthogonal projections e and $p^{\otimes m}$, such that e is a trivial one-dimensional projection and $p^{\otimes m}$ is (equivalent to)

$$p \otimes p \otimes \cdots \otimes p \in M_{2^m}(\mathcal{C}((S^2)^m)).$$

Put

$$m(1) = m(2) = 1, \quad m(j) = 2^{j-2}, \quad j \geq 3,$$

and put

$$B_j = (e + p^{\otimes m(j)})M_{2^{m(j)+1}}(\mathcal{C}((S^2)^{m(j)}))(e + p^{\otimes m(j)}), \quad A = \bigotimes_{j=1}^{\infty} B_j.$$

As in Example 4.7 (again using Lemma 1.8) we see that A is a $\mathcal{C}(X)$ -algebra with fibres A_x isomorphic to the CAR-algebra for all $x \in X$.

For each $n \geq 2$ consider the two projections

$$\begin{aligned} f_n &= e \otimes e \otimes 1_{B_3} \otimes \cdots \otimes 1_{B_n} \\ &= e \otimes e \otimes (e + p^{\otimes m(3)}) \otimes \cdots \otimes (e + p^{\otimes m(n)}), \\ g_n &= (p \otimes e + e \otimes p) \otimes 1_{B_3} \otimes \cdots \otimes 1_{B_n} \\ &= (p \otimes e + e \otimes p) \otimes (e + p^{\otimes m(3)}) \otimes \cdots \otimes (e + p^{\otimes m(n)}) \end{aligned}$$

in $\bigotimes_{j=1}^n B_j$. Let f and g be the corresponding projections in A . We show below that $4[f] \leq 3[g]$ and that $[f] \not\leq [g]$ in $V(A)$. This will settle the claims made in the first paragraph of this example.

The projection $f_2 \oplus f_2 \oplus f_2 \oplus f_2$ has dimension 4, the projection $g_2 \oplus g_2 \oplus g_2$ has dimension 6, and as the difference $6 - 4 = 2$ is greater than or equal to $3/2 = (\dim((S^2)^2) - 1)/2$ it follows from [17, 9.1.2] that $f_2 \oplus f_2 \oplus f_2 \oplus f_2$ is equivalent to a subprojection of $g_2 \oplus g_2 \oplus g_2$. This entails that $f \oplus f \oplus f \oplus f \preceq g \oplus g \oplus g$, whence $4[f] \leq 3[g]$ in $V(A)$.

To show that $f \not\leq g$ it suffices to show that $f_n \not\leq g_n$ for all $n \geq 2$. We show that the Euler class of g_n is non-zero for all n . Hence g_n cannot dominate any trivial projection, and as f_n does dominate a trivial projection, f_n is not equivalent to a subprojection of g_n .

To calculate the Euler class of g_n let us first note that g_n belongs to a matrix algebra over

$$\bigotimes_{j=1}^n \mathcal{C}((S^2)^{m(j)}) \cong \bigotimes_{j=1}^{M(n)} \mathcal{C}(S^2) \cong \mathcal{C}((S^2)^{M(n)}),$$

where $M(j) = m(1) + m(2) + \cdots + m(j)$. Expand g_n as follows:

$$\begin{aligned} g_n &= (p \otimes e + e \otimes p) \otimes (e + p^{\otimes m(3)}) \otimes \cdots \otimes (e + p^{\otimes m(n)}) \\ &= \sum_{I \subseteq \{3,4,\dots,n\}} p \otimes e \otimes q_I + \sum_{I \subseteq \{3,4,\dots,n\}} e \otimes p \otimes q_I \end{aligned}$$

where

$$q_I = r_3 \otimes r_4 \otimes \cdots \otimes r_n, \quad r_i = \begin{cases} p^{\otimes m(i)}, & i \in I \\ e, & i \notin I. \end{cases}$$

Following the notation of [31], for each subset J of $\{1, 2, \dots, M(n)\}$ put

$$p_J = r_1 \otimes r_2 \otimes \cdots \otimes r_{M(n)}, \quad r_j = \begin{cases} p, & j \in J, \\ e, & j \notin J, \end{cases}$$

which belongs to a matrix algebra over $\mathcal{C}((S^2)^{M(n)}) \cong \bigotimes_{j=1}^{M(n)} \mathcal{C}(S^2)$.

For $3 \leq j \leq n$ let H_j be the set of integers i such that $M(j-1) + 1 \leq i \leq M(j)$, and for $I \subseteq \{3, 4, \dots, n\}$ put

$$J_1(I) = \{1\} \cup \bigcup_{i \in I} H_i, \quad J_2(I) = \{2\} \cup \bigcup_{i \in I} H_i.$$

Then

$$p \otimes e \otimes q_I \sim p_{J_1(I)}, \quad e \otimes p \otimes q_I \sim p_{J_2(I)},$$

whence

$$g_n \sim \bigoplus_{I \subseteq \{3,4,\dots,n\}} p_{J_1(I)} \oplus \bigoplus_{I \subseteq \{3,4,\dots,n\}} p_{J_2(I)}.$$

We can now use [31, Proposition 3.2 and Lemma 4.1] to conclude that the Euler class of g_n is non-zero. We have to show that the family

$$\{J_1(I) \mid I \subseteq \{3, 4, \dots, n\}\} \cup \{J_2(I) \mid I \subseteq \{3, 4, \dots, n\}\}$$

admits a matching (cf. [31, Proposition 3.2 (iii)]). Now, $J_1(\emptyset) = \{1\}$ and $J_2(\emptyset) = \{2\}$, so it is clear how to match these two sets. If $I \subseteq \{3, 4, \dots, n\}$ is non-empty, then choose the matching elements for $J_1(I)$ and for $J_2(I)$ in the set $H_{\max I}$. This is possible because there are 2^{k-3} subsets I of $\{3, 4, \dots, n\}$ with $\max I = k$ and there are $2 \cdot 2^{k-3} = m(k)$ elements in H_k .

4.9 We have just seen that Theorem 4.6 does not remain true for infinite dimensional X . However, we shall show in Proposition 4.11 that when \mathcal{D} -stability of the fibres is replaced by ‘local’ \mathcal{D} -stability, a statement similar to 4.6 holds. We first show that \mathcal{D} -stability passes to pullbacks.

PROPOSITION: *Let \mathcal{D} be a K_1 -injective strongly self-absorbing C^* -algebra. Suppose we have a pullback diagram of separable C^* -algebras*

$$\begin{array}{ccc} C & \xrightarrow{\pi_1} & A_1 \\ \pi_2 \downarrow & & \downarrow \varphi_1 \\ A_2 & \xrightarrow{\varphi_2} & B \end{array}$$

where C is the pullback of (A_1, A_2) along (φ_1, φ_2) , and at least one of the maps φ_1, φ_2 is surjective. If A_1 and A_2 are \mathcal{D} -stable, then so is C .

PROOF: Let us assume that φ_2 is surjective. Therefore π_1 is also surjective, and so, C is an extension of A_1 by $\ker(\pi_1)$. We may identify

$$C = \{(a_1, a_2) \in A_1 \oplus A_2 \mid \varphi_1(a_1) = \varphi_2(a_2)\} \subseteq A_1 \oplus A_2.$$

Under this identification, we have $\ker(\pi_1) = \{(0, a) \mid \varphi_2(a) = 0\} = 0 \oplus \ker(\varphi_2)$. Since A_2 is \mathcal{D} -stable, and \mathcal{D} -stability passes to ideals, it follows that $\ker(\varphi_2)$, and therefore $\ker(\pi_1)$, are \mathcal{D} -stable. Since A_1 and $\ker(\pi_1)$ are \mathcal{D} -stable, and \mathcal{D} -stability passes to extensions by [34, Theorem 4.3], it follows that C is \mathcal{D} -stable, as required. \square

4.10 REMARK: In Proposition 4.9, we assumed that at least one of the maps into B is surjective. This assumption cannot be removed. To see that, consider the following pullback diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda \mapsto \lambda 1} & \mathcal{D} \\ \lambda \mapsto \lambda 1 \downarrow & & \downarrow d \mapsto d \otimes 1 \\ \mathcal{D} & \xrightarrow{d \mapsto 1 \otimes d} & \mathcal{D} \otimes \mathcal{D} \end{array}$$

Even though the two copies of \mathcal{D} are of course \mathcal{D} -stable, \mathbb{C} is not.

4.11 PROPOSITION: *Let \mathcal{D} be a K_1 -injective strongly self-absorbing C^* -algebra. Let X be a locally compact metrizable space (not necessarily of finite dimension) and A a separable $C_0(X)$ -algebra. Suppose each $x \in X$ has a compact neighborhood $V_x \subset X$ such that A_{V_x} is \mathcal{D} -stable for each $x \in X$. Then, A is \mathcal{D} -stable.*

PROOF: By 1.9 it suffices to prove the assertion for compact X . But then we may assume that X is covered by finitely many closed subsets $V_i, i = 1, \dots, k$, such that each A_{V_i} is \mathcal{D} -stable. Set

$$K_\ell := \bigcup_{i=1}^{\ell} V_i.$$

We prove that A_{K_ℓ} is \mathcal{D} -stable for each $\ell = 1, \dots, k$ by induction; this will suffice as $X = K_k$. $A_{K_1} = A_{V_1}$ is \mathcal{D} -stable by assumption. Suppose now that $A_{K_{\ell-1}}$ is \mathcal{D} -stable for some $\ell \in \{2, \dots, k\}$. We may write A_{K_ℓ} as a pullback

$$\begin{array}{ccc} A_{K_\ell} & \longrightarrow & A_{K_{\ell-1}} \\ \downarrow & & \downarrow \\ A_{V_\ell} & \longrightarrow & A_{K_{\ell-1} \cup V_\ell}, \end{array}$$

where the maps are the restriction maps, and in particular are surjective. By Proposition 4.9, we see that A_{K_ℓ} is \mathcal{D} -stable, as required. \square

4.12 REMARK: Recall ([3]) that a separable C^* -algebra A is said to be *approximately divisible* if there is a central sequence of unital embeddings of $M_2 \oplus M_3$ into $\mathcal{M}(A)$. Approximate divisibility is seen as a certain regularity property, which is weaker than \mathcal{D} -stability for \mathcal{D} a UHF algebra, and stronger than \mathcal{Z} -stability. One might ask if similar results to the ones we have obtained can be found for approximate divisibility. However, the following example shows that the analogues of both Theorem 4.6 and Proposition 4.11 fail in this case.

Denote by $M_{2^\infty}, M_{3^\infty}$ the UHF algebras of types $2^\infty, 3^\infty$ respectively. Let

$$A = \{f \in \mathcal{C}([0, 1], M_{2^\infty} \otimes M_{3^\infty}) \mid f(0) \in M_{2^\infty} \otimes 1, f(1) \in 1 \otimes M_{3^\infty}\}$$

The embedding of $\mathcal{C}([0, 1])$ into A as the scalar functions gives A the structure of a $\mathcal{C}([0, 1])$ -algebra. The fibres are M_{2^∞} at 0, M_{3^∞} at 1, and $M_{2^\infty} \otimes M_{3^\infty}$ elsewhere, so in particular, each fibre is approximately divisible. Furthermore, $A_{[0, 2/3]}$ and $A_{[1/3, 1]}$ are approximately divisible (they absorb M_{2^∞} and M_{3^∞} , respectively). Thus, A satisfies analogous conditions to those of both Theorem 4.6 and Proposition 4.11, when one replaces \mathcal{D} -stability by approximate divisibility. However, it is easy to see that A is unital with no nontrivial projections, and hence cannot be approximately divisible.

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DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE'ER SHEVA 84105, ISRAEL

E-mail address: ilan@math.bgu.ac.il

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF SOUTHERN DENMARK, CAMPUSVEJ 55, DK-5230 ODENSE M, DENMARK

E-mail address: mikael@imada.sdu.dk

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNSTER, EINSTEINSTR. 62, D-48149 MÜNSTER, GERMANY

E-mail address: wwinter@math.uni-muenster.de