

CONTINUOUS FAMILIES OF E_0 -SEMIGROUPS

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ABSTRACT. We consider families of E_0 -semigroups continuously parametrized by a compact Hausdorff space, which are cocycle-equivalent to a given E_0 -semigroup β . When the gauge group of β is a Lie group, we establish a correspondence between such families and principal bundles whose structure group is the gauge group of β .

Let \mathcal{H} be a Hilbert space, which we will always assume to be separable and infinite-dimensional, and let $\mathfrak{B}(\mathcal{H})$ denote the $*$ -algebra of all bounded operators over \mathcal{H} . An E_0 -semigroup acting on $\mathfrak{B}(\mathcal{H})$ is a point- σ -weakly continuous family $\beta = \{\beta_t : \mathfrak{B}(\mathcal{H}) \rightarrow \mathfrak{B}(\mathcal{H})\}_{t \geq 0}$ of unital $*$ -endomorphisms such that $\beta_0 = \text{id}$. We direct the reader to [Arv03] for a general reference on the theory of E_0 -semigroups.

This paper studies continuous families of E_0 -semigroups parametrized by a compact Hausdorff space, where the E_0 -semigroups are all cocycle equivalent to a given E_0 -semigroup β . Suitable notions of continuity and equivalence are introduced below. If the gauge group G of β is a Lie group, we show that such continuous families are classified by principal G -bundles. Gauge groups of E_0 -semigroups were computed by Arveson in [Arv89] for the type I case. In the type II case, the gauge groups were computed recently for several classes of examples by Alevras, Powers and Price in [APP06] and by Jankowski and the second author in [JM11]. Indeed, in many of the known examples the gauge group is a Lie group. Specializing to the case of E_0 -semigroups of type I, one can recast those principal bundles as vector bundle invariants. The case of continuous families of single endomorphisms of $\mathfrak{B}(\mathcal{H})$ (of finite index) was studied in [Hir04], where it was shown that such families of endomorphisms are classified by vector bundle invariants of dimension given by the index of the endomorphism. We thus obtain an analogy between the case of continuous families of endomorphisms and continuous families of E_0 -semigroups.

In the case of families of one-parameter automorphism groups, the gauge group is \mathbb{R} , hence the principal bundle invariants are trivial. We treat this case separately, since we establish this triviality result under (a priori) weaker continuity assumptions, using techniques from [Bar54]. This corresponds to a parametrized version of Wigner's theorem.

Given an E_0 -semigroup β acting on $\mathfrak{B}(\mathcal{H})$ we will say that a strongly continuous family of unitary operators $\{U_t \in \mathcal{U}(\mathcal{H}) : t \geq 0\}$ is a β -cocycle if $U_0 = 1$ and $U_{t+s} = U_t \beta_t(U_s)$ for all $t, s \geq 0$. We emphasize that in this paper all cocycles will be unitary cocycles. We will denote by C_β the set of all β -cocycles. An E_0 -semigroup α is cocycle equivalent to β if there exists a β -cocycle U_t such that $\alpha_t(X) = U_t \beta_t(X) U_t^*$ for all $t \geq 0$ and $X \in \mathfrak{B}(\mathcal{H})$. We will denote by \mathcal{E}_β be the set of all E_0 -semigroups acting on $\mathfrak{B}(\mathcal{H})$ which are cocycle equivalent to the E_0 -semigroup β .

If \mathcal{H} is separable, then the unitary group $\mathcal{U}(\mathcal{H})$ is a Polish group when endowed with the relative strong operator topology. Recall that a Polish space is a topological space with a separable completely metrizable topology, and a Polish group is a topological group whose topology is Polish. Let \mathbb{R}_+ denote the half-open interval $[0, \infty)$. Let us endow $C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$

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with pointwise multiplication and the compact open topology. Since $\mathcal{U}(\mathcal{H})$ is Polish, we have that $C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$ is a Polish group (this can be proven using Theorem 4.19 of [Kec95]).

Let β be an E_0 -semigroup acting on $\mathfrak{B}(\mathcal{H})$. Every β -cocycle $\{U_t : t \in \mathbb{R}_+\}$ can be seen as an element of $C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$, and we will always endow the set C_β with the relative topology in $C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$. Since C_β is clearly closed in $C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$, it follows that it is also a Polish space.

Recall that the gauge group G_β is the set of local (unitary) β -cocycles: $G_\beta = \{U \in C_\beta : U_t \beta_t(A) = \beta_t(A) U_t, \forall A \in \mathfrak{B}(\mathcal{H}), \forall t \geq 0\}$. Notice that G_β is a closed subgroup of $C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$, hence it is a Polish group.

There is a natural (right) action of G_β on $C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$ given by $(U \cdot Q)_t = U_t Q_t$ for $U \in C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$, $Q \in G_\beta$. Furthermore, C_β is invariant under this action. Let $\pi : C_\beta \rightarrow \mathcal{E}_\beta$ be given by $\pi(U) = \alpha$ where $\alpha_t(A) = U_t \beta_t(A) U_t^*$ for all $A \in \mathfrak{B}(\mathcal{H})$ and $t \geq 0$. One readily checks that the fibers of π are exactly the orbits of the G_β action.

We may thus endow \mathcal{E}_β with the quotient topology of C_β/G_β . If \mathfrak{X} is a compact Hausdorff space, a continuous map $\alpha : \mathfrak{X} \rightarrow \mathcal{E}_\beta$ will be said to be a *continuous family of E_0 -semigroups parametrized by \mathfrak{X}* in the cocycle-equivalence class of β . We shall also denote the set of all such continuous families by $\mathcal{E}_\beta(\mathfrak{X})$. Given $\alpha \in \mathcal{E}_\beta(\mathfrak{X})$, we shall denote by α^x the E_0 -semigroup at the point $x \in \mathfrak{X}$.

Let \mathfrak{X} be a compact Hausdorff space. If $\theta \in \mathcal{E}_\beta(\mathfrak{X})$ is a continuous family of E_0 -semigroups, we shall say that a family $\{U^x : x \in \mathfrak{X}\}$ is a *continuous family of θ -cocycles* if U^x is an θ^x -cocycle for each $x \in \mathfrak{X}$ and the map $x \mapsto U^x$ is continuous as a map $\mathfrak{X} \rightarrow C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$. We shall say that two continuous parametrized families $\theta, \sigma \in \mathcal{E}_\beta(\mathfrak{X})$ are *equivalent* if there exists a continuous family of σ -cocycles U such that $\theta_t^x(A) = U_t^x \sigma_t^x(A) U_t^{x*}$ for all $t \in \mathbb{R}_+$, $x \in \mathfrak{X}$ and $A \in \mathfrak{B}(\mathcal{H})$ (this is clearly an equivalence relation). Note that for any two families $\theta, \sigma \in \mathcal{E}_\beta(\mathfrak{X})$ and for every $x \in \mathfrak{X}$ it is possible to choose a θ^x -cocycle U^x such that $\sigma_x(A) = U_t^x \theta_t^x(A) U_t^{x*}$ for all $t \geq 0$ and $A \in \mathfrak{B}(\mathcal{H})$, however it may not be possible to choose the cocycles in a continuous way with respect to x . We will say that a continuous family $\theta \in \mathcal{E}_\beta(\mathfrak{X})$ is *trivializable* if it is equivalent to the constant family equal to β for all $x \in \mathfrak{X}$.

We note that there are other natural notions of continuity for families of E_0 -semigroups. We will say that the family of E_0 -semigroups $\{\alpha^x \in \mathcal{E}_\beta : x \in \mathfrak{X}\}$ is *σ -weakly continuous* if for any $A \in \mathfrak{B}(\mathcal{H})$, the map $\mathfrak{X} \times \mathbb{R}_+ \rightarrow \mathfrak{B}(\mathcal{H})$ given by $(x, t) \mapsto \alpha_t^x(A)$ is σ -weakly continuous. It is straightforward to check that any continuous family of E_0 -semigroups is also σ -weakly continuous, however we do not know whether the converse holds. We note (Remark 2.6) however, that for one-parameter automorphism groups the two notions do coincide, and thus it is conceivable that they coincide for E_0 -semigroups as well.

1. PRELIMINARIES

In this section we recall a few results concerning G -bundles.

Let G be a locally compact group with identity e . A G -space is a topological space \mathfrak{X} with a fixed continuous right action of G (all our G -spaces will be right G -spaces). We will say that G acts *freely* on \mathfrak{X} if for every $g \in G$, $g \neq e$ and for every $x \in \mathfrak{X}$, $xg \neq x$.

Given a free action of G on \mathfrak{X} , let $R = \{(x, y) : \exists g \in G, xg = y\}$ denote the corresponding orbit equivalence relation endowed with the relative topology in $\mathfrak{X} \times \mathfrak{X}$. Since G acts freely, given a pair $(x, y) \in R$ there exists a unique element $\Delta(x, y) \in G$ such that $y = x\Delta(x, y)$.

A free action of G on a space \mathfrak{X} determines a G -bundle $\xi = (\mathfrak{X}, \pi, \mathfrak{X}/G)$ where $\pi : \mathfrak{X} \rightarrow \mathfrak{X}/G$ is the quotient map. We will say that ξ is a *principal G -bundle* if the map $\Delta : R \rightarrow G$ described above is continuous (we follow the terminology of [Hus94]; this is called a Cartan principal G -bundle in [Pal61]).

We emphasize that a principal G -bundle is not assumed to be locally trivial. Under additional assumptions on G and \mathfrak{X} , however, local triviality is automatic.

Theorem 1.1 ([Pal61], p.315). *Let G be a Lie group acting freely on a completely regular topological space \mathfrak{X} , and let $\xi = (\mathfrak{X}, \pi, \mathfrak{X}/G)$ be the corresponding G -bundle. If ξ is a principal G -bundle then it is locally trivial.*

We will say that a continuous action of a locally compact group G on a completely regular space X makes it a *proper G -space* if every point $x \in \mathfrak{X}$ has a neighborhood U_x which is *small* in the following sense: each $y \in \mathfrak{X}$ has a neighborhood V_y such that $\{g \in G : U_x \cap (V_y \cdot g) \neq \emptyset\}$ has compact closure in G . We recall the following.

Proposition 1.2 ([Pal61], Proposition 1.2.8). *If \mathfrak{X} is proper G -space, then \mathfrak{X}/G is completely regular.*

An open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ for a topological space B is *numerable* provided that there exists a locally finite partition of unity $\{f_\lambda\}_{\lambda \in \Lambda}$ such that $f_\lambda^{-1}((0, 1]) \subseteq U_\lambda$ for all λ .

A principal G -bundle $\xi = (E, p, B)$ is *numerable* provided that there exists a numerable cover $\{U_\lambda\}_{\lambda \in \Lambda}$ for B such that the restriction bundle $\xi|_{U_\lambda}$ is trivial for all $\lambda \in \Lambda$.

In particular, if $\xi = (E, p, B)$ is a locally trivial principal G -bundle and B is paracompact, then ξ is numerable. We remark that a pullback of a numerable principal G -bundle is also numerable. For the following definition, see Chapter 4, Section 10 of [Hus94].

Definition 1.3. A numerable principal G -bundle $\xi_0 = (E_0, p_0, B_0)$ is *universal* if and only if

- (1) For each numerable principal G -bundle $\xi = (E, p, B)$, there exists a continuous map $f : B \rightarrow B_0$ such that ξ and the pull-back $f^*(\xi_0)$ are isomorphic;
- (2) If $f, g : B \rightarrow B_0$ are two continuous maps such that $f^*(\xi_0)$ and $g^*(\xi_0)$ are isomorphic, then f and g are homotopic.

The following result provides an effective criterion for establishing that a numerable principal bundle is universal.

Theorem 1.4 ([Dol63], Theorem 7.5). *A numerable principal G -bundle $\xi = (E, p, B)$ is universal if and only if E is contractible.*

2. CONTINUOUS FAMILIES OF AUTOMORPHISM GROUPS

The purpose of this section is to show that any σ -weakly continuous family of one-parameter automorphism groups is trivialisable. In effect, this amounts to a version of Wigner's theorem with a parameter. This can be obtained by modifying Bargmann's proof of Wigner's theorem from [Bar54].

Definition 2.1. Let G be an abelian topological group and let $0 \in \mathcal{N} \subseteq \mathbb{R}$ be an open interval. We say that a continuous map $\omega : \mathcal{N} \times \mathcal{N} \rightarrow G$ is a *local G -valued multiplier* on \mathcal{N} if

$$(1) \quad \omega(t, 0) = \omega(0, t) = e_G$$

$$(2) \quad \omega(t, s)\omega(t + s, r) = \omega(s, r)\omega(t, s + r)$$

for all $r, s, t \in \mathcal{N}$ such that the required sums are also in \mathcal{N} . If G is a Lie group, we will say that a G -multiplier ω is *smooth* if it is a C^∞ function. Given a topological space \mathfrak{X} and a Lie group G , we will say that a family $\{\omega^x : x \in \mathfrak{X}\}$ of G -multipliers on \mathcal{N} is continuous if the map from $\mathfrak{X} \times \mathcal{N} \times \mathcal{N}$ to G given by $(x, t, s) \mapsto \omega^x(t, s)$ is continuous. We will say that the family is *smooth* if ω^x is smooth for every $x \in \mathfrak{X}$ and the map $(x, t, s) \mapsto \partial_t^n \partial_s^m \omega^x(t, s)$ is continuous, for any $n, m \in \mathbb{Z}_+$.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We will refer to local \mathbb{T} -valued multipliers simply as *local multipliers*, and we will refer to \mathbb{R} -valued multipliers as *additive local multipliers*.

Lemma 2.2. *Let \mathfrak{X} be a compact Hausdorff space and let $\{\alpha^x : x \in \mathfrak{X}\}$ be a family of $*$ -automorphisms of $\mathfrak{B}(\mathcal{H})$ such that for all $A \in \mathfrak{B}(\mathcal{H})$, the map $(x, t) \mapsto \alpha_t^x(A)$ is σ -weakly continuous. Then there exists an open interval $0 \in \mathcal{N} \subseteq \mathbb{R}$ and a family of unitaries $\{U_t^x : t \in \mathcal{N}, x \in \mathfrak{X}\}$ which is jointly strongly continuous in (x, t) and which satisfies*

$$(3) \quad \alpha_t^x(A) = U_t^x A U_t^{x*}, \quad \forall A \in \mathfrak{B}(\mathcal{H}), \forall t \in \mathcal{N}, \forall x \in \mathfrak{X}.$$

Furthermore, there exists a unique continuous family of local multipliers $\{\omega^x : x \in \mathfrak{X}\}$ on \mathcal{N} which satisfies

$$U_t^x U_s^x = \omega^x(t, s) U_{t+s}^x$$

for all $x \in \mathfrak{X}$ and $t, s \in \mathcal{N}$ such that $t + s \in \mathcal{N}$. Moreover, by shrinking \mathcal{N} if necessary, $\sigma^x = -i \operatorname{Log} \omega^x$ (the principal branch of \log) is a well-defined bounded local additive multiplier on \mathcal{N} and $\{\sigma^x : x \in \mathfrak{X}\}$ is a continuous family of additive multipliers on \mathcal{N} .

Proof. Let $\xi \in \mathcal{H}$ be a unit vector, and let p be the minimal projection onto the space spanned by ξ . Since $\alpha_t^x(p)$ is a strongly jointly continuous family of minimal projections and $\alpha_0^x(p) = p$, notice that $\{(x, t) \in \mathfrak{X} \times \mathbb{R} : \|\alpha_t^x(p)\xi\| > 1/2\}$ is an open set containing $\mathfrak{X} \times \{0\}$. Thus, by compactness, there exists $\delta > 0$ such that $\|\alpha_t^x(p)\xi\| > 1/2$ for all $x \in \mathfrak{X}$, $|t| < \delta$. By normalization we obtain a norm-continuous family of normalized vectors $\eta_t^x \in \mathcal{H}$ such that $\alpha_t^x(p)\eta_t^x = \eta_t^x$ for all $x \in \mathfrak{X}$ and $|t| < \delta$. Now define for all $x \in \mathfrak{X}$, $|t| < \delta$, the map $U_t^x : \mathcal{H} \rightarrow \mathcal{H}$ given by

$$U_t^x A \xi = \alpha_t(A) \eta_t^x$$

A direct computation shows that this map is a well-defined isometry, which is surjective hence unitary, and that equation (3) holds for all $A \in \mathfrak{B}(\mathcal{H})$ and $(x, t) \in \mathfrak{X} \times (-\delta, \delta)$. We observe that this family is weakly continuous since for all $t, s \in (-\delta, \delta)$, $x, y \in \mathfrak{X}$, $A \in \mathfrak{B}(\mathcal{H})$ and $\psi \in \mathcal{H}$,

$$\begin{aligned} | \langle (U_t^x - U_s^y) A \xi, \psi \rangle | &= | \langle \alpha_t^x(A) \eta_t^x - \alpha_s^y(A) \eta_s^y, \psi \rangle | \\ &\leq | \langle \alpha_t^x(A) \eta_t^x - \alpha_t^x(A) \eta_s^y, \psi \rangle | + | \langle \alpha_t^x(A) \eta_s^y - \alpha_s^y(A) \eta_s^y, \psi \rangle | \\ &\leq \|A\| \| \eta_t^x - \eta_s^y \| \| \psi \| + | \langle \alpha_t^x(A) \eta_s^y - \alpha_s^y(A) \eta_s^y, \psi \rangle | \end{aligned}$$

and we are assuming that $\alpha_s^y(A)$ is σ -weakly continuous on s, y . Hence it is also strongly continuous since both topologies coincide on the unitary group. The verification of the remaining assertions of the lemma is straightforward so it will be omitted. \square

The next two lemmas and the following theorem are a straightforward modification of Bargmann's work [Bar54] on multipliers for the context of families with a parameter. For the reader's convenience we provide full details. Although we chose to focus on the case of actions of \mathbb{R} on $\mathfrak{B}(\mathcal{H})$ by automorphisms, it is clear that analogous versions of the following results also hold for actions of Lie groups, by a similar modification of Bargmann's method. In the following, given a set $S \subseteq \mathbb{R}$, we will denote $S + S = \{t + s : t, s \in S\}$.

Lemma 2.3. *Suppose that \mathfrak{X} is a compact Hausdorff space. If $\{\sigma^x : x \in \mathfrak{X}\}$ is a continuous family of bounded local additive multipliers on an open interval $0 \in \mathcal{N} \subseteq \mathbb{R}$, then there exists a smooth family $\{\zeta^x : x \in \mathfrak{X}\}$ of bounded local additive multipliers on an open interval $0 \in \mathcal{N}' \subseteq \mathcal{N}$ and a continuous function $\phi : \mathfrak{X} \times (\mathcal{N}' + \mathcal{N}') \rightarrow \mathbb{R}$ such that for all $x \in \mathfrak{X}$ and $t, s \in \mathcal{N}'$,*

$$(4) \quad \zeta^x(t, s) = \sigma^x(t, s) - \phi(x, t + s) + \phi(x, t) + \phi(x, s), \quad \text{and} \quad \phi(x, 0) = 0.$$

Proof. Let $0 \in \mathcal{N}_1 \subseteq \mathcal{N}$ be an open interval such that $\mathcal{N}_1 + \mathcal{N}_1 \subseteq \mathcal{N}$, and let $0 \in \mathcal{N}' \subseteq \mathcal{N}_1$ be an open interval such that $\mathcal{N}' + \mathcal{N}' \subseteq \mathcal{N}_1$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative compactly

supported C^∞ function whose support is contained in \mathcal{N}' and moreover $\int_{\mathbb{R}} f(x)dx = 1$. Now define a function $a : \mathfrak{X} \times \mathcal{N} \rightarrow \mathbb{R}$ by

$$a(x, t) = - \int_{\mathbb{R}} \sigma^x(t, r) f(r) dr.$$

(The integrand vanishes outside \mathcal{N}'). Then we have that $a(x, t)$ is continuous. Moreover, we have that the family $\{\theta^x : x \in \mathfrak{X}\}$ given by

$$\theta^x(t, s) = \sigma^x(t, s) - a(x, t + s) + a(x, t) + a(x, s), \quad \forall t, s \in \mathcal{N}_1$$

is a continuous family of additive multipliers on \mathcal{N}_1 since $a(x, 0) = 0$ for all $x \in \mathfrak{X}$. Furthermore, notice that by using the multiplier property (2),

$$\begin{aligned} \theta^x(t, s) &= \int_{\mathbb{R}} [\sigma^x(t, s) + \sigma^x(t + s, r) - \sigma^x(t, r) - \sigma^x(s, r)] f(r) dr \\ &= \int_{\mathbb{R}} [\sigma^x(t, s + r) - \sigma^x(t, r)] f(r) dr \\ &= \int_{\mathbb{R}} \sigma^x(t, u) [f(u - s) - f(u)] du. \end{aligned}$$

It follows that $\theta^x(t, s)$ is infinitely differentiable with respect to s . Now define $b : \mathfrak{X} \times \mathcal{N}_1 \rightarrow \mathbb{R}$ by

$$b(x, t) = - \int_{\mathbb{R}} \theta^x(r, t) f(r) dr.$$

Then we have that $b(x, t)$ is continuous and $b(x, 0) = 0$ for all $x \in \mathfrak{X}$. By defining

$$\zeta^x(t, s) = \theta^x(t, s) - b(x, t + s) + b(x, t) + b(x, s), \quad \forall t, s \in \mathcal{N}'$$

we again obtain a continuous family $\{\zeta^x : x \in \mathfrak{X}\}$ of additive multipliers on \mathcal{N}' and by computations analogous to those carried out above, we obtain for all $t, s \in \mathcal{N}'$,

$$\zeta^x(t, s) = \int_{\mathbb{R}} \theta^x(u, s) [f(u - t) - f(u)] du.$$

Thus for every $x \in \mathfrak{X}$, ζ^x is a C^∞ function, and it is clear that it constitutes a smooth family. Finally, note that (4) is satisfied for $\phi(x, t) = a(x, t) + b(x, t)$. \square

Lemma 2.4. *Let \mathfrak{X} be a compact Hausdorff space. If $\{\zeta^x : x \in \mathfrak{X}\}$ is a continuous family of bounded local additive multipliers on an open interval $0 \in \mathcal{N} \subseteq \mathbb{R}$, then there exists an open interval $0 \subseteq \mathcal{N}' \subseteq \mathcal{N}$ and a continuous function $\phi : \mathfrak{X} \times \mathcal{N} \rightarrow \mathbb{R}$ such that for all $x \in \mathfrak{X}$ and $t, s \in \mathcal{N}'$,*

$$\zeta^x(t, s) = \phi(x, t + s) - \phi(x, t) - \phi(x, s), \quad \text{and} \quad \phi(x, 0) = 0.$$

Proof. By the previous lemma, it suffices to prove the statement in the case that $\{\zeta^x : x \in \mathfrak{X}\}$ is a smooth family on \mathcal{N} . Let us consider

$$\theta(x, t, s) = \partial_s \zeta^x(t, s)$$

which is a continuous function of (x, t, s) since the family is smooth. Let $0 \in \mathcal{N}'$ be an open interval such that $\mathcal{N}' + \mathcal{N}' \subseteq \mathcal{N}$. By differentiating the additive identity for ζ^x corresponding to multiplier identity (2) with respect to r at $r = 0$, we obtain for all $x \in \mathfrak{X}$ and $t, s \in \mathcal{N}'$,

$$(5) \quad \theta(x, t + s, 0) = \theta(x, s, 0) + \theta(x, t, s).$$

Define $\phi : \mathfrak{X} \times \mathcal{N} \rightarrow \mathbb{R}$ by

$$\phi(x, t) = \int_0^t \theta(x, u, 0) du$$

It is clear that ϕ is continuous and $\phi(x, 0) = 0$ for all $x \in \mathfrak{X}$. Moreover, for all $u, v \in \mathcal{N}'$,

$$\begin{aligned} \phi(x, u+v) - \phi(x, u) - \phi(x, v) &= \int_u^{u+v} \theta(x, r, 0) dr - \int_0^v \theta(x, s, 0) ds \\ &= \int_0^v \theta(x, u+s, 0) du - \int_0^s \theta(x, s, 0) ds \\ \text{by (5)} &= \int_0^v \theta(x, u, s) ds \\ &= \int_0^v \partial_s \zeta^x(u, s) ds \\ &= \zeta^x(u, v) - \zeta^x(u, 0) \\ \text{by (1)} &= \zeta^x(u, v). \end{aligned}$$

□

We now obtain a version of Wigner's theorem with a parameter.

Theorem 2.5. *Let \mathfrak{X} be a compact Hausdorff space and let $\{\alpha^x : x \in \mathfrak{X}\}$ be a family of $*$ -automorphisms of $\mathfrak{B}(\mathcal{H})$ such that for all $A \in \mathfrak{B}(\mathcal{H})$, the map $(x, t) \mapsto \alpha_t^x(A)$ is σ -weakly continuous. Then there exists a family of unitaries $\{U_t^x : t \in \mathbb{R}, x \in \mathfrak{X}\}$ in $\mathfrak{B}(\mathcal{H})$ such that*

- (1) $(x, t) \mapsto U_t^x$ is strongly continuous
- (2) for each $x \in \mathfrak{X}$, $\{U_t^x : t \in \mathbb{R}\}$ is a unitary group
- (3) $\alpha_t^x(A) = U_t^x A U_t^{x*}$, for all $A \in \mathfrak{B}(\mathcal{H})$, $t \in \mathbb{R}$ and $x \in \mathfrak{X}$.

In particular, the family $\{\alpha^x : x \in \mathfrak{X}\}$ is trivializable.

Proof. By Lemma 2.2, there exists an open interval $0 \in \mathcal{N}$ and a family $\{V_t^x : t \in \mathcal{N}, x \in \mathfrak{X}\}$ implementing the family $\{\alpha^x : x \in \mathfrak{X}\}$, and a continuous family of additive multipliers $\{\sigma^x : x \in \mathfrak{X}\}$ on \mathcal{N} such that

$$V_t^x V_s^x = \exp(-i\sigma^x(t, s)) V_{t+s}^x,$$

for all $t, s \in \mathcal{N}$ such that $t+s \in \mathcal{N}$ and $x \in \mathfrak{X}$. Furthermore, by Lemmas 2.3 and 2.4, perhaps by shrinking \mathcal{N} , there exists a continuous function $\phi : \mathfrak{X} \times \mathcal{N} \rightarrow \mathbb{R}$ and an open interval $0 \in \mathcal{N}' \subseteq \mathcal{N}$ such that for all $x \in \mathfrak{X}$ and $t, s \in \mathcal{N}'$,

$$\sigma^x(t, s) = \phi(x, t+s) - \phi(x, t) - \phi(x, s), \quad \text{and} \quad \phi(x, 0) = 0.$$

In particular, if we define $U_t^x = e^{-i\phi(x, t)} V_t^x$ for $x \in \mathfrak{X}$ and $t \in \mathcal{N}$ we obtain a new family of unitary operators which is strongly continuous in (x, t) , which implements the automorphisms $\{\alpha^x : x \in \mathfrak{X}\}$ for $t \in \mathcal{N}$, and which furthermore satisfies

$$U_t^x U_s^x = U_{t+s}^x, \quad \forall t, s \in \mathcal{N}', \forall x \in \mathfrak{X}.$$

It is clear that we now wish to define for all $t \in \mathbb{R}$, $U_t^x = (U_{t/n}^x)^n$ where $n \in \mathbb{N}$ is large enough so that $t/n \in \mathcal{N}'$. This gives rise to a well-defined family of operators, because for all $x \in \mathfrak{X}$ and $t_1, \dots, t_m, s_1, \dots, s_n \in \mathcal{N}'$ such that $\sum_{j=1}^n t_j = \sum_{k=1}^m s_k$, it is easy to see that

$$U_{t_1}^x U_{t_2}^x \dots U_{t_n}^x = U_{s_1}^x U_{s_2}^x \dots U_{s_m}^x.$$

The strong continuity of the family $\{U_t^x\}$ follows from the continuity of ϕ and the family V_t^x on (x, t) . □

Despite the foregoing result, by itself the notion of point- σ -weak continuity for a family of E_0 -semigroups $\{\alpha^x : x \in \mathfrak{X}\}$ of $\mathfrak{B}(\mathcal{H})$ over a compact Hausdorff space \mathfrak{X} is not sufficiently stringent for applications. Indeed, cocycle-conjugacy classes need not be preserved under continuous parametrizations in this sense. For an example of this phenomenon, let β be any

E_0 -semigroup acting on $\mathfrak{B}(\mathcal{H})$ which is not cocycle conjugate to an automorphism group, and consider the family $\alpha = \{\alpha^x : x \in [0, 1]\}$ where

$$\alpha_t^x(A) := \beta_{xt}(A), \quad \forall A \in \mathfrak{B}(\mathcal{H}), t \geq 0.$$

It is clear that α^x is an E_0 -semigroup for every $x \in [0, 1]$, and for every $A \in \mathfrak{B}(\mathcal{H})$ the map $(x, t) \mapsto \alpha_t^x(A)$ is σ -weakly continuous. Notice however that $\alpha_t^0(x) = x$ for all $t > 0$ and $x \in \mathfrak{B}(\mathcal{H})$.

In this context, it seems that the appropriate reading of Theorem 2.5 is that it provides a lift of a continuous family of E_0 -semigroups which are *in the same cocycle-cojugacy class*.

Remark 2.6. *We note that it follows immediately from Theorem 2.5 that any σ -weakly continuous family of automorphism groups is a continuous family in our sense.*

3. CONTINUOUS FAMILIES OF E_0 -SEMIGROUPS

We first recall the following theorem concerning invariant metrics. For a proof, see Theorem 8.3, p.70 of [HR79].

Theorem 3.1 (Birkhoff-Kakutani). *Any metrizable topological group admits a left-invariant metric.*

The Polish group of interest to us here is the group $C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$. We note that we can write such a metric explicitly as follows. Let $(\psi_k)_{k \in \mathbb{N}}$ be a dense sequence in the unit sphere of \mathcal{H} . For any two elements $U, V \in C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$, define

$$d(U, V) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sup_{t \in [0, k]} \|U_t \psi_k - V_t \psi_k\|$$

The exact form of the metric is of no importance in the sequel, however.

Proposition 3.2. *Let H be a Polish group, and let G be a subgroup of H . If X is any subset of H which is G -invariant, then G acts freely and continuously on X . If in addition G is locally compact (in the relative topology), then X is a proper (right) G -space and $(X, \pi, X/G)$ is a principal G -bundle.*

Proof. It is clear that G acts continuously and freely on H , hence it acts continuously and freely on X . Now suppose that it is locally compact.

Let d be a left invariant metric on H . We denote by $B_r(h)$ the ball with center h and radius r in H . Given $x \in X$, we denote $B_r^X(x) = B_r(x) \cap X$. Since G is locally compact, it follows that there is some $r_0 > 0$ such that for any $0 < r < r_0$, any set of diameter $4r$ in G has compact closure in G , and in particular $B_{2r}(h) \cap G$ has compact closure in G for any $h \in H$. Fix such an $0 < r < r_0$.

We prove that X is a proper G -space by showing that for every $x \in X$ the neighborhood $B_r^X(x)$ is small. Fix $y \in X$, and let V_y be an open neighborhood of y in X such that for all $z \in V_y$, $d(z^{-1}x, y^{-1}x) < r$. This is possible by continuity of the operations in H . We now claim that the set

$$A_{x,y} = \{g \in G : B_r^X(x) \cap V_y \cdot g \neq \emptyset\}$$

has a compact closure in G . To see that, we note that $g \in A_{x,y}$ if and only if there exists $z \in V_y$ such that $zg \in B_r^X(x)$, so $d(g, z^{-1}x) = d(zg, x) < r$ by left-invariance. Thus, $d(g, y^{-1}x) < 2r$, so $A_{x,y} \subseteq B_{2r}(y^{-1}x) \cap G$, and therefore has compact closure in G .

Since the action of G on X is free, in order to prove that X is a principal G -bundle it remains to show that the function Δ uniquely defined on the equivalence relation $R = \{(x, y) \in X \times X : \exists g \in G, y = xg\}$ by

$$x\Delta(x, y) = y, \quad \forall (x, y) \in R$$

is continuous. But this is evident since $\Delta(x, y) = x^{-1}y$. \square

We obtain the following immediate corollary by taking $H = C(\mathbb{R}_+, \mathcal{U}(\mathcal{H}))$, $X = C_\beta$ and $G = G_\beta$.

Corollary 3.3. *If β is an E_0 -semigroup acting on $\mathfrak{B}(\mathcal{H})$, then its gauge group G_β acts freely on the cocycle space C_β via right multiplication: for all $U \in C_\beta$ and $Q \in G_\beta$, $(U \cdot Q)_t := U_t Q_t$, $t \geq 0$. Furthermore, if G_β is locally compact, then C_β is a proper G_β -space and $\Xi_\beta = (C_\beta, \pi, \mathcal{E}_\beta)$ is a principal G_β -bundle, where $\pi : C_\beta \rightarrow \mathcal{E}_\beta$ denotes the natural quotient map onto \mathcal{E}_β endowed with the quotient topology.*

Proposition 3.4. *If β is an E_0 -semigroup acting on $\mathfrak{B}(\mathcal{H})$, then C_β is contractible.*

Proof. Consider the E_0 -semigroup β' on $\mathfrak{B}(\mathcal{H} \otimes L^2(\mathbb{R}_+))$ given by $\beta' = \beta \otimes id_{\mathfrak{B}(L^2(\mathbb{R}_+))}$. Since β is cocycle conjugate to β' , it suffices to show that $C_{\beta'}$ is contractible. Let $\{V_\lambda : \lambda \geq 0\}$ be the right shift semigroup on $L^2(\mathbb{R}_+)$ and let $S_\lambda = 1_{\mathfrak{B}(\mathcal{H})} \otimes V_\lambda$, $\lambda \geq 0$ be the right shift semigroup on $\mathcal{H} \otimes L^2(\mathbb{R}_+)$. Denote $P_\lambda = 1 - S_\lambda S_\lambda^* = 1 \otimes (1 - V_\lambda V_\lambda^*)$ for all $\lambda \geq 0$. Observe that for all $t \geq 0$ and $\lambda \geq 0$, $\beta'_t(S_\lambda) = S_\lambda$ and $\beta'_t(P_\lambda) = P_\lambda$.

Let us define $F : [0, \infty) \times C_{\beta'} \rightarrow C_{\beta'}$ by $F(\lambda, U)_t = P_\lambda + S_\lambda U_t S_\lambda^*$. We need to check that the range of F is in $C_{\beta'}$ (once that is done, it is clear that F is continuous). Given a β' -cocycle U , we must show that for all $\lambda \geq 0$, $F(\lambda, U)$ is also a cocycle, i.e. we should check that $F(\lambda, U)_t \beta'_t(F(\lambda, U)_s) = F(\lambda, U)_{t+s}$, and that $F(\lambda, U)$ is unitary. Indeed,

$$\begin{aligned} F(\lambda, U)_t \beta'_t(F(\lambda, U)_s) &= (P_\lambda + S_\lambda U_t S_\lambda^*) \beta'_t(P_\lambda + S_\lambda U_s S_\lambda^*) \\ &= (P_\lambda + S_\lambda U_t S_\lambda^*) (P_\lambda + S_\lambda \beta'_t(U_s) S_\lambda^*) \\ &= P_\lambda + S_\lambda U_t \beta'_t(U_s) S_\lambda^* = P_\lambda + S_\lambda U_{t+s} S_\lambda^* \\ &= F(\lambda, U)_{t+s} \end{aligned}$$

thus $F(\lambda, U)$ satisfies the cocycle condition. To check that it is unitary, we write

$$F(\lambda, U)_t F(\lambda, U)_t^* = (P_\lambda + S_\lambda U_t S_\lambda^*) (P_\lambda + S_\lambda U_t^* S_\lambda^*) = P_\lambda + S_\lambda U_t U_t^* S_\lambda^* = 1$$

and checking that $F(\lambda, U)_t^* F(\lambda, U)_t = 1$ is similar. Now note that for any sequence (λ_n, U_n) , such that $\lambda_n \rightarrow \infty$, we have that $F(\lambda_n, U)$ converges to 1 as $n \rightarrow \infty$, hence we have a well-defined continuous extension of F to $[0, \infty] \times C_{\beta'}$ which is a contraction as required. \square

This allows us to prove the following theorem.

Theorem 3.5. *Let β be an E_0 -semigroup such that its gauge group G_β is a Lie group. Then $\Xi_\beta = (C_\beta, \pi, \mathcal{E}_\beta)$ is a universal principal G_β -bundle.*

Proof. In view of Corollary 3.3 and Proposition 3.4, in order to conclude that the bundle Ξ_β is universal, by Theorem 1.4 it only remains to show that Ξ_β is numerable. Since C_β is completely regular (it is Polish) and G_β is a Lie group, we have that Ξ_β is locally trivial by Theorem 1.1. Thus to complete the proof it suffices to show that \mathcal{E}_β is paracompact. We will show that \mathcal{E}_β is second countable and regular, hence metrizable by Urysohn's metrization theorem, and metrizability implies paracompactness. It follows from Proposition 1.2 and Corollary 3.3 that \mathcal{E}_β is completely regular. Thus it remains to establish that it is second countable. It suffices to show that the quotient map $\pi : C_\beta \rightarrow \mathcal{E}_\beta$ is open, since C_β is second countable, as it is in fact Polish. That π is open is a consequence of the local triviality of the bundle, as follows. Let $\mathcal{O} \subseteq C_\beta$ be an open subset. Let $U \in \mathcal{O}$ be a given cocycle. It suffices to show that the point $\pi(U)$ has a neighborhood which is contained in $\pi(\mathcal{O})$. By local triviality, $\pi(U)$ has a neighborhood \mathcal{N} such that $\pi^{-1}(\mathcal{N})$ is homeomorphic to $\mathcal{N} \times G_\beta$ via a homeomorphism which conjugates π to the first coordinate projection. In particular, the restriction of π to $\pi^{-1}(\mathcal{N})$ is open. Thus, $\pi(\mathcal{O} \cap \pi^{-1}(\mathcal{N}))$ is an open neighborhood of U contained in $\pi(\mathcal{O})$, as required. \square

Thus for continuous families of E_0 -semigroups in the cocycle conjugacy class of β , we obtain the following.

Corollary 3.6. *Let β be an E_0 -semigroup whose gauge group is a Lie group. Let \mathfrak{X} be a compact Hausdorff space. For every continuous family $\alpha \in \mathcal{E}_\beta$, define the locally trivial principal G_β -bundle over \mathfrak{X} given by the pullback $\xi(\alpha) := \alpha^*(\Xi_\beta)$. For all $\alpha, \gamma \in \mathcal{E}_\beta(\mathfrak{X})$,*

- (1) $\xi(\alpha) \cong \xi(\gamma)$ if and only if α is equivalent to γ .
- (2) $\xi(\alpha)$ is trivial if and only if the family α is trivializable.
- (3) Any locally trivial principal G_β -bundle over \mathfrak{X} is isomorphic to $\xi(\alpha)$ for some $\alpha \in \mathcal{E}_\beta(\mathfrak{X})$.

Proof. For (1), it is clear that if α and γ are equivalent then $\xi(\alpha) \cong \xi(\gamma)$. For the converse, let $j : \xi(\alpha) \rightarrow \xi(\gamma)$ be an isomorphism. We define a continuous family of γ -cocycles $\{V^x : x \in \mathfrak{X}\}$ as follows: for each $t \geq 0$, $x \in \mathfrak{X}$, let U_t^x be any element in the fiber of $\xi(\alpha)$ over x . We let

$$V_t^x = U_t^x j(U_t^x)^*$$

We note that V_t^x does not depend on the choice of U_t^x , because j is G -equivariant: if W_t^x is another element of the fiber of $\xi(\alpha)$ over x , then there exists $Q \in G_\beta$ such that $W_t^x = U_t^x Q_t$ and $W_t^x j(W_t^x)^* = U_t^x Q_t j(U_t^x Q_t)^* = U_t^x Q_t Q_t^* j(U_t^x)^* = V_t^x$. It is straightforward to verify that V^x is a γ^x -cocycle. Continuity now follows from local triviality of $\xi(\alpha)$.

Part (2) follows immediately from item (1), and part (3) follows directly from universality of the bundle Ξ_β . \square

We remark that, in general, the gauge group of an E_0 -semigroup need not be locally compact. For example, let us consider E_0 -semigroups of type I.

Let \mathcal{K} be a separable Hilbert space, which is allowed to have dimension zero. We consider the Heisenberg group $\text{Heis}(\mathcal{K}) = \{(c, \xi) : c \in \mathbb{R}, \xi \in \mathcal{K}\}$ with the product topology and the operations

$$\begin{aligned} (c_1, \xi_1) \cdot (c_2, \xi_2) &= (c_1 + c_2 + \text{Im}\langle \xi_1, \xi_2 \rangle, \xi_1 + \xi_2) \\ (c, \xi)^{-1} &= (-c, -\xi) \end{aligned}$$

The unitary group $\mathcal{U}(\mathcal{K})$ acts on $\text{Heis}(\mathcal{K})$ by automorphisms as follows,

$$U \cdot (c, \xi) = (c, U\xi)$$

hence we have a semidirect product $\text{Heis}(\mathcal{K}) \rtimes \mathcal{U}(\mathcal{K})$. This is a Polish group with the product topology (here $\mathcal{U}(\mathcal{K})$ has the strong operator topology as usual). Its operations can be described explicitly by the following:

$$\begin{aligned} (c_1, \xi_1, U_1) \cdot (c_2, \xi_2, U_2) &= (c_1 + c_2 + \text{Im}\langle \xi_1, U_1 \xi_2 \rangle, \xi_1 + U_1 \xi_2, U_1 U_2) \\ (c, \xi, U)^{-1} &= (-c, -U^{-1} \xi, U^{-1}) \end{aligned}$$

Naturally, when $\dim \mathcal{K} = 0$, we mean $\text{Heis}(\mathcal{K}) \rtimes \mathcal{U}(\mathcal{K}) = \mathbb{R}$.

Theorem 3.7 ([Arv89]). *Let β be an E_0 -semigroup of type I, and let \mathcal{K} be a Hilbert space such that $\dim \mathcal{K} = \text{ind } \beta$. Then G_β is isomorphic as a topological group to $\text{Heis}(\mathcal{K}) \rtimes \mathcal{U}(\mathcal{K})$.*

It follows that in the case $\text{ind } \beta = \infty$ the gauge group is manifestly not locally compact. Nevertheless, Theorem 3.7 implies that when the gauge group of an E_0 -semigroup of type I is locally compact then it is a Lie group. That is also the case for many of the E_0 -semigroups of type II_0 analyzed recently by [APP06] and [JM11]. In fact, to our knowledge in all cases when the gauge group has been proven to be locally compact, it has also turned out to be a Lie group.

Suppose $\dim(\mathcal{K}) = n < \infty$. Since $\text{Heis}(\mathcal{K})$ is contractible, $\text{Heis}(\mathcal{K}) \times \mathcal{U}(\mathcal{K})$ -bundles are prolongations of $\mathcal{U}(\mathcal{K})$ -bundles (see Chapter 6 of [Hus94]). In turn there is a functorial bijection between $\mathcal{U}(\mathcal{K})$ -bundles and the corresponding associated n -dimensional (complex) vector bundles. Thus for the case of continuous families of E_0 -semigroups of type I_n , Corollary 3.6 is equivalent to the following form. Note the analogy with Theorem 1 of [Hir04].

Corollary 3.8. *Suppose that β is an E_0 -semigroup of type I_n , and let \mathfrak{X} be a compact Hausdorff space. To each continuous family $\alpha \in \mathcal{E}_\beta(\mathfrak{X})$ we have an associated n -dimensional vector bundle $v(\alpha)$ over \mathfrak{X} such that the following hold. For all $\alpha, \gamma \in \mathcal{E}_\beta(\mathfrak{X})$,*

- (1) $v(\alpha) \cong v(\gamma)$ if and only if α is equivalent to γ .
- (2) $v(\alpha)$ is trivial if and only if the family α is trivializable.
- (3) Any n -dimensional vector bundle over \mathfrak{X} is isomorphic to $v(\alpha)$ for some $\alpha \in \mathcal{E}_\beta(\mathfrak{X})$.

We note that if G_β is a Lie group, then it follows that a σ -weakly continuous family is a continuous family if and only if it is locally trivial. We have seen that local triviality (indeed, triviality) is automatic in the case of automorphism groups, but we do not know if this is the case for E_0 -semigroups. However, it follows from the corollaries above that there exist topological obstructions for having a parametrized version of Wigner's theorem in the σ -weakly continuous setting. For instance, we have the following.

Corollary 3.9. *If \mathfrak{X} is a compact Hausdorff space which admits a non-trivial n -dimensional complex vector bundle, then there exists a (locally trivial) σ -weakly continuous family of E_0 -semigroups of type I_n parametrized by \mathfrak{X} which is not trivializable.*

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