From cycle complex constructions to Voevodsky motives
in the
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ISHAI DAN-COHEN

0.1. At this point we have an abelian category of mixed motives which we constructed out of Voevodsky's triangulated category. It has the following formal properties.

(1) It's Tannakian.
(2) There's a special invertible object $\mathbb{Q}(1)$. Its tensor powers $\mathbb{Q}(i)$ are simple, and nonisomorphic.
(3) Every simple object is isomorphic to a $\mathbb{Q}(i)$.
(4) $\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(i)) = 0$ for $i \leq 0$.

A construction known as "mixed Tate duality" produces a graded Hopf algebra A^{mot}.

0.2. The main point of this talk is that there's an alterna-A, A^{BK}, which is constructed directly out of Bloch's cylce complex N; the two may be compared.

0.3. The comparison goes thru an auxiliary gadget, the *derived category of cell modules*, a triangulated category constructed directly out of certain dg N-modules. The construction of a functor

$$D^b(\mathrm{Corep}^{\mathrm{gr}} A) \to D\mathrm{Cell}(N)$$

is general nonsense. When N is Bloch's cycle dga, we may construct a further functor

$$D\mathrm{Cell}(N) \to DM_{\mathrm{gm}}(k)\,.$$

1. Mixed Tate duality

1.1. Fix a field k of char. zero. A *mixed Tate category* over k is a Tannakian category equipped with a special invertible object $k(1)$ such that the objects $k(i) := k(1)^{\otimes i}$ are mutually nonisomorphic simple objects, every simple object is isomorphic to one of them, and

$$\mathrm{Ext}^1(k(0), k(n)) = 0$$

for $n \leq 0$.

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1.2. Let's use the phrase *graded prounipotent k-group* to refer to a prounipotent group $U = \operatorname{Spec} A$ equipped with an $\mathbb{N}$-grading of A such that $A_0 = k$. Mixed Tate duality sets up a correspondence between mixed Tate categories over k and graded prounipotent k-groups. In one direction we simply associate to a graded prounipotent group the category of finite dimensional graded representations

$$\mathbf{Rep}^{\mathrm{gr}}\, U\,.$$

To go in the other direction, we can construct a certain canonical fiber functor, and then study its fundamental group. Briefly,

$$\omega(M) = \bigoplus \operatorname{Hom}(k(-n), \operatorname{gr}^W_{2n} M)$$

and we find that

$$\underline{\operatorname{Aut}}^{\otimes}(\omega) = \mathbb{G}_m \ltimes U\,.$$

with U graded prounipotent. But there's also a more direct construction, which is the main focus of this section.

1.3. Let T be a mixed Tate category over k. An *n-framed object* (M, v, f) is an object M plus nonzero morphisms

$$v : k(0) \to \operatorname{gr}^W_0 M \qquad \text{and} \qquad f : \operatorname{gr}^W_{-2n} M \to k(n)$$

We consider the equivalence relation generated by $M \sim M'$ if there exists a map $M \to M'$ commuting with the frame and we let A_n denote the set of equivalence classes. Addition is defined by

$$[M, v, f] + [M', v', f'] = [M \oplus M', (v, v'), f + f']\,.$$

This gives A_n the structure of an abelian group with neutral element is $k(0) \oplus k(n)$. Multiplication

$$A_k \otimes A_l \to A_{k+l}$$

is induced by the tensor product. The unit and counit are given by the inclusion of and projection onto

$$A_0 = k\,.$$

The comultiplication is given by

$$\Delta = \bigoplus_{0 \le p \le n} \Delta_{p,n-p} : A_n \to A_p \otimes A_{n-p}$$

with

$$\Delta_{p,n-p}[M, v, f] = \sum [M, v, b_i^\vee] \otimes [M, b_i, f](-p)$$

for any basis b of

$$\operatorname{Hom}(k(p), \operatorname{gr}^W_{-2p} M)\,.$$

The antipode is then uniquely determined.

1.4. Example. The category of mixed Tate Voevodsky motives with $\mathbb{Q}$ coefficients is a mixed Tate category, so there's an associated graded prounipotent group

$$U^{\mathrm{mot}} = \operatorname{Spec} A^{\mathrm{mot}}\,.$$

2. From representations to cell modules

2.1. An Adams graded cdga consists of a unital, bi-graded k-algebra

$$N = \bigoplus_{m\in\mathbb{Z}, r\in\mathbb{N}} N^m(r)$$

together with a vector space endomorphism d of degree 1 for the $\mathbb{Z}$-grading and degree 0 for the $\mathbb{N}$-grading. The $\mathbb{Z}$-grading is called the *cohomological grading*, while the $\mathbb{N}$-grading is called the *Adams grading*. We require that N be graded-commutative for the cohomological grading, that $d^2 = 0$, and that

$$d(ab) = (da)b + (-1)^{\deg a} adb\,.$$

2.2. Suppose $N(0) = k$ concentrated in cohomological degree 0, and let

$$N^+ = N(\geq 1)\,.$$

We let $(BN)^{\bullet,\bullet}$ denote the double complex with nonzero terms given by

$$(BN)^{0,0} = k \qquad \text{and} \qquad (BN)^{n,m} := ((N^+)^{\otimes(-m)})^n \text{ for } m < 0\,.$$

A decomposable element $x_1 \otimes \cdots x_m$ is denoted by $[x_1|\cdots|x_m]$. With this traditional notation, the second differential $(BN)^{n,m} \to (BN)^{n,m+1}$ is given by

$$d[x_1|\cdots|x_m] = \sum_j (-1)^j [x_1|\cdots|x_j x_{j+1}|\cdots|x_m]\,.$$

We let

$$BN := \mathrm{Tot}(BN)^{\bullet,\bullet}\,.$$

We define the *shuffle product* by

$$[x_1|\cdots|x_m] \cup [x_{m+1}|\cdots|x_{m+n}] = \frac{m!n!}{(m+n)!} \sum_\sigma \mathrm{sgn}(\sigma)[x_{\sigma(1)}|\cdots|x_{\sigma(n+m)}]$$

where σ ranges over all permutations with $\sigma(1) < \cdots < \sigma(n)$ and $\sigma(n+1) < \cdots < \sigma(n+m)$. We define a coproduct

$$\delta : BN \to BN \otimes BN$$

by $\delta[x_1|\cdots|x_m] = \sum_{i=0}^m [x_1|\cdots|x_i] \otimes [x_{i+1}|\cdots|x_m]$, with empty tensor equal to 1.

Proposition. Let $A := H^0 BN$. Then $\mathrm{Spec}\, A$ is a graded prounipotent k-group.

2.3. An Adams graded *dg* N-module consists of a $\mathbb{Z} \times \mathbb{Z}$-graded vector space

$$M = \bigoplus_{n,r\in\mathbb{Z}} M^n(r)$$

plus an endomorphism d, and a module structure

$$N \otimes M \to M\,.$$

The superscript grading is called the *cohomological grading*; the parenthetical one is called the *Adams grading*. The differential d is required to be graded of cohomological degree 1 and Adams degree 0. The module structure is required to be graded of degree zero. Finally, we require $d^2 = 0$ and

$$d(am) = (da)m + (1)^{\deg a} am\,.$$

M is a *cell module* if the underlying bi-graded module is free of finite rank. Adams graded dg A-modules admit internal homs and tensor products. Given a morphism

$$f : M \to M'$$

there's a notion of cone:

$$Cone(f)^n(r) = M'^n(r) \oplus M^{n+1}(r)$$

with differential $d(n,m) = (dn + fm, -dm)$, and a cone sequence

$$M \xrightarrow{f} M' \to \text{Cone}\, f \to M[1]$$

where $M[1]^n(r) = M^{n+1}(r)$ with differential $-d$. The *derived category of cell modules* is obtained simply by setting

$$\text{Hom}(M, M') = h^0 \,\text{hom}(M, M')\,.$$

2.4. We assume from now on that N is *cohomologically connected*; this means that $H^0 = k$ and $H^i = 0$ for $i < 0$. The goal of this section is to outline the construction of a functor

$$\rho : D^b(\text{Corep}^{\text{gr}} A) \to D\text{Cell}(N)$$

from the bounded derived category of graded corepresentations to the derived category of cell modules. Here's an outline. We let

$$\wedge^* V \to N$$

denote the 1-*minimal model.* It has the following properties:

1. The underlying algebra is a wedge algebra.
2. There's an increasing filtration F of V with $F_0 = V$, $\bigcup F_i V = V$ and
$$dF_i V \subset \wedge^2 F_{i-1} V\,.$$
3. $H^1 \wedge^* V \to H^1 N$ is iso.
4. $H^2 \wedge^* V \to H^2 N$ is injective.

The piece of the differential

$$d : V \to \wedge^2 V$$

gives V the structure of a graded Lie coalgebra. We construct a sequence of maps as follows.

$$D^b(\text{Corep}^{\text{gr}} A) \xleftarrow{\psi_1} D^b(\text{Corep}^{\text{gr}} H^0 B \wedge^* V) \xrightarrow{\psi_2} D^b(\text{Corep}^{\text{gr}} V) \xrightarrow{\psi_3} D\text{Cell}(\wedge^* V)) \xrightarrow{\psi_4} D\text{Cell}(N)\,.$$

Moreover, we will claim that ψ_1 is an equivalence.

Actually, we fist need to replace N by a *connected* cdga, that is, one with $N^0 = k$ and $N^i = 0$ for $i < 0$. For this, we have the following

Proposition. A quasi-isomorphism of Adams graded dga's induces an isomorphism of graded Hopf algebras, as well as an equivalence of derived categories of cell modules.

2.5. The 1-minimal model of N is constructed as follows. We let $V_1 = H^1 N$ and we let

$$\phi : V_1 \to N^1$$

be a lifting. We endow $\wedge^* V_1$ with the structure of a cdga by setting

$$d = 0\,.$$

Then ϕ extends to a map

$$\phi : \wedge^* V_1 \to N$$

of cdga's. Next we let V_2 be the kernel of H^2:

$$0 \to V_2 \to \wedge^2 V_1 \to H^2 N$$

be a lifting of H^1, let $\wedge^* V_1$ denote the graded commutative symmetric algebra with dga structure given by setting $d = 0$. There's an induced map

$$\phi_1 : \wedge^* V_1 \to A\,.$$

The inclusion $V_2 \to \wedge^2 V_1$ gives $\wedge^*(V_1 \oplus V_2)$ the structure of a 1-minimal dga. To extend ϕ to a map

$$\phi : \wedge^*(V_1 \oplus V_2) \to N\,,$$

we choose a lifting of the map

$$V_2 \to B^2 N$$

to N^1. Continuing in this way, we obtain the 1-minimal model, with

$$V := \bigoplus V_i\,.$$

Proposition. If N is connected, then the map

$$H^0 B \wedge^* V \to A$$

is an isomorphism.

This completes our discussion of ψ_1.

2.6. For ψ_2 we note that $(B \wedge^* V)^{-1} = 0$. Indeed, the relevant diagonals of the bar double complex look like so.

$$\begin{array}{ccccccccc}
 & & & & & & & & \iddots \\
 & & & & 0 & \longrightarrow & V^{\otimes 3} & \longrightarrow & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & V^{\otimes 2} & \longrightarrow & V \otimes \wedge^2 V \;\oplus\; \wedge^2 V \otimes V & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & V & \longrightarrow & \wedge^2 V & & & & \\
\downarrow & & \downarrow & & & & & & \\
k & \longrightarrow & 0 & & & & & &
\end{array}$$

So there's a map

$$H^0 B \wedge^* V \subset TV \twoheadrightarrow V\,.$$

Given a comodule $E \to (H^0 B \wedge^* V) \otimes E$, we claim that the composite

$$E \to (H^0 B \wedge^* V) \otimes E \to V \otimes E$$

makes E into a (graded) corepresentation of V.

2.7. For ψ_3, given a comodule E we let $\psi_3(E) = (\wedge^* V) \otimes E$ and use the Leibniz rule to extend d. We take total complexes to obtain a functor

$$C^b(\mathrm{Corep}^{\mathrm{gr}} V) \to \mathrm{Cell}(\wedge^* V)\,,$$

and we claim that this descends to a functor between derived categories.

2.8. The functors $\psi_{1,2,3}$ are all equivalences. By contrast, for ψ_4 we have

Theorem. The functor ψ_4 is an equivalence if and only if the map $\wedge^* V \to N$ is a quasi-isomorphism.

3. The cycle complex and the journey to Voevodsky motives

3.1. A *cubical face* of $\mathbb{A}^n$ is a closed subscheme defined by a subset of the family of equations

$$t_i = \epsilon \qquad \text{for } i = 1, \dots, n \text{ and } \epsilon \in \{0, 1\}\,.$$

Let

$$\iota_{j,\epsilon} : \mathbb{A}^{n-1} \to \mathbb{A}^n$$

be the closed immersion which inserts an ϵ in the j^{th} coordinate. For $r \geq 0$, we define a complex $N(r)$ of $\mathbb{Q}$ vector spaces as follows. $N(0)$ is just $\mathbb{Q}$ in degree zero. For $r > 0$, $N(r)^m$ is the space of alternating cycles of codimension r on $\mathbb{A}^{2r-m}$ which intersect every cubical face properly. The boundary map

$$d : N(r)^m \to N(r)^{m+1}$$

is given by

$$\sum_{j=1}^{m} (-1)^{j-1} (\iota_{j,1}^* - \iota_{j,0}^*)\,.$$

The advantage of using cubes over simplices is that there's a natural product structure, making N into an Adams graded cdga. Bloch's higher Chern character is an isomorphism

$$H^m(N(r)) = K_{2r-m}(k)^{(r)}\,.$$

In particular, if k is a number field, then Borel's computations show that N is quasi-isomorphic to its 1-minimal model.

3.2. The category of finite dimensional graded corepresentations of H^0BN is what Bloch-Kriz define to be the category of mixed Tate motives. The goal for this section is to outline the construction of a functor, allegedly an equivalence, from its bounded derived category to the triangulated category of Voevodsky.

3.3. Recall that $\mathbf{SmCor}(k)$ is the category whose objects are the smooth k-schemes and whose morphisms are finite correspondences. A *Nisnevich cover* of X is an étale cover $U \to X$ such that for any finite field extension k' of k, the map

$$U(k') \to X(k')$$

is surjective. We let $\mathbf{Sh}^{\mathrm{Nis}}(\mathbf{SmCor}(k))$ denote the category of presheaves on $\mathbf{SmCor}(k)$ satisfying the sheaf condition with respect to Nisnevich covers. The category $\mathbf{DM}_-^{\mathrm{eff}}(k)$ is the full subcategory of $D^-(\mathbf{Sh}^{\mathrm{Nis}}\mathbf{SmCor}(k))$ consisting roughly of those complexes whose cohomology sheaves have homotopy-invariant Nisnevich cohomology presheaves.

The *homological motive functor*

$$m : \mathbf{Sm}(k) \to DM_-^{\mathrm{eff}}(k)$$

sends X to the Nisnevich sheafification of the complex

$$\cdots \to z(X \times \mathbb{A}^n) \to z(X \times \mathbb{A}^{n-1}) \to \cdots \to z(X)$$

where z denotes the group of cycles. The boundary maps may be defined by using simplices or cubes. Voevodsky's theorem is that m extends to an embedding

$$DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \subset DM_-^{\mathrm{eff}}(k)\,.$$

3.4. We construct an Adams graded cdga object N^{gm} in $C^{-}(\mathbf{Sh}^{\mathrm{Nis}}\mathbf{SmCor}(k))$ as follows. Let $\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^r)$ denote the Nisnevich sheaf with transfers associated to $\mathbb{A}^r$. Given $X \in \mathbf{SmCor}(k)$, we consider the complex

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^r)(X \times \mathbb{A}^n)/\mathrm{degn} \to \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^r)(X \times \mathbb{A}^{n-1})/\mathrm{degn} \to \cdots \to \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^r)(X)$$

where degn refers to cycles that arise by pullback along a projection. Then

$$N^{\mathrm{gm}}(r)^m(X) \subset \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^r)(X \times \mathbb{A}^{2r-m})$$

is the subgroup of cycles which are alternating for the cation of S_{2r-m} and symmetric for the action of S_r.

3.5. Lemma. The inclusion and the pullback along the projection

$$\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^r)(\mathbb{A}^{2r-m}) \subset z^r(\mathbb{A}^{2r-m} \times \mathbb{A}^r) \xleftarrow{\pi^*} z^r(\mathbb{A}^{2r-m})$$

induce a quasi-isomorphism

$$N^{\mathrm{gm}}(k) \cong N\,.$$

3.6. Given a cell $N^{\mathrm{gm}}(k)$-module M, fix a basis $\{m_j\}$, with m_j in Adams degree r_j and cohomological degree r_j, and write

$$dm_j = \sum_i a_{i,j} m_j \qquad (*)$$

We associate to M the object of $DM^{\mathrm{eff}}_{-}(k)$ given by

$$\bigoplus_j N^{\mathrm{gm}}(r_j)[n_j]m_j$$

with differential given by the same formula $(*)$. Similarly, if $f : M \to M'$ is a morphism of cell modules, we fix bases $\{m_j\}$, $\{m'_j\}$, we write

$$f(m_j) = \sum f_{i,j} m'_i\,,$$

and we use the same formula to define a map of associated motives.

3.7. Theorem. The above construction sinduce an isomorphism

$$A^{BK} \to A^{\mathrm{mot}}\,.$$

Fakultät für Mathematik
Universität Duisburg-Essen
Universitätstr. 2
45117 Essen
Germany
E-mail address: ishaidc@gmail.com