Divisors and their intersections on wonderful compactifications

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ISHAI DAN-COHEN

INTRODUCTION

Let $G$ be a semisimple and simply connected algebraic group over $k$, $\sigma$ an automorphism of order 2, $H = G^\sigma$. The main point of this talk is to explain the following yoga: If you want to understand line bundles and intersection multiplicities on $G/N(H)$, you should wonderfully compactify and go to the closed orbit.

Section 1 will be about line bundles in the case $G \times G$, $\sigma(g,g') = (g',g)$. Section 2) Line bundles and intersection multiplicities for the case $SL_n$, $\sigma(A) = tA^{-1}$. This is harder, and we give fewer details.

Section 3) application to counting quadrics, a very superficial sketch.

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Let $\Delta \subset X^*(T)$ be the set of roots (= nonzero weights for the action of $T$ on $g$.

Let $\Delta^+$ denote the set of positive roots (those occurring in $n = \text{Lie} U$, $U$ the unipotent radical of $B$).

Let $\{\alpha_1^\vee, \alpha_i^\vee\} \subset X_*(T)$ denote the simple coroots,

$\{\chi_1, \chi_i\} \subset X^*(T)$ the dual basis of “fundamental weights”.

Fix a regular, dominant weight $\lambda$. Recall that this means that $\lambda = \sum n_i \chi_i$ with $n_i \neq 0$ for all $i$.

Let $M = M(\lambda)$ the associated simple representation of highest weight $\lambda$.

Consider the action of $G \times G$ on $\text{End} M$ by $(g,g')(\phi) = g\phi g'^{-1}$.

Let $h$ denote the identity element,

$[h]$ its image in $\mathbb{P}^\vee \text{End} M = \text{Proj} S^\bullet (\text{End} M)^\vee = \text{lines}!$

Let $X = (G \times G)[h]$.

Let $h = \sum h_\mu$ be the decomposition of $h$ for the action of $1 \times T$. Then actually

$h = m_\lambda^* \otimes m_\lambda + \sum_{\mu < \lambda} h_\mu$

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for $m_\lambda \in M_\lambda$, $m^*_\lambda \in M^*_\lambda$, with $\langle m^*_\lambda, m_\lambda \rangle = 1$, meaning that $\lambda$ has multiplicity one and the other $\mu$’s are smaller than $\lambda$.

Let $X$ denote the open subset cut out by $m_\lambda \otimes m^*_\lambda \in (\text{End } M)^*$. If we use $(-\alpha_1, -\alpha_l)$ to embed $T_{\text{ad}}$ in $\mathbb{A}^l$ then the map

$$T \to (T \times T)[h]$$

given by

$$t \mapsto (1, t)[h] = \lambda(t) \sum_{n_1, n_l \in \mathbb{N}} \alpha_1(t^{-n_1}) \cdots \alpha_l(t^{-n_l}) h_{\lambda - \sum n_i \alpha_i}$$

extends to a closed immersion $\mathbb{A}^l \to X$, which extends to an isomorphism

$$U \times \mathbb{A}^l \times U^- \xrightarrow{\sim} X.$$ 

In $X_0$ we have the boundary divisor

$$U \times \mathbb{A}^l \times U^- \setminus U \times \mathbb{G}_m^I \times U^- = X_1 + +X_l$$

with closure

$$X \setminus (G \times G)[h] = X_1 + +X_l.$$ 

Then the closed orbit is given by

$$Y = \bigcap X_i = G/B \times G/B.$$ 

1.2. Divisors coming from Bruhat decomp. Let $W = N(T)/T$, denote by $s_i$ the elt corresp to $\alpha_i$, and by $w_0$ the longest element which interchanges $\Delta^+$ with $\Delta^-$. Let $D_i = B_{\text{ad}} s_i w_0 B_{\text{ad}}$.

Then

(1) Each $D_i$ is a divisor. Let’s call these theBruhat divisors.

(2) $X \setminus X = D_1 w_0 + +D_l w_0$

(3) Pic $X$ is freely generated by the $D_i$.

Proof. (1) Recall that $G_{\text{ad}}$ is stratified by subsets $B_{\text{ad}} w B_{\text{ad}}$, $w \in W$, and that there are isomorphisms

$$\mathbb{G}_a^{\Delta^+ \cap w(\Delta^-)} \times B_{\text{ad}} \xrightarrow{\sim} B_{\text{ad}} w B_{\text{ad}}$$

So $B_{\text{ad}} w B_{\text{ad}}$ is a divisor precisely when $w = s_i w_0$.

(2) We’ve seen that every orbit meets $X$, so

$$\overline{G_{\text{ad}}} \setminus X = X \setminus X.$$ 

On the other hand, we have

$$G_{\text{ad}} \cap X = U T_{\text{ad}} U^-$$

$$= B_{\text{ad}} B_{\text{ad}}^\perp$$

$$= B_{\text{ad}} w_0 B_{\text{ad}} w_0$$

is the translate of the open Bruhat cell by $w_0$. So $G_{\text{ad}} \setminus X$ is a sum of $w_0$-translates of Bruhat divisors.
(3) Since \( \text{Pic} X = 0 \), \( \mathbb{Z}(D_1, \ldots, D_l) \rightarrow \text{Pic} X \). Suppose \( \sum a_i D_i = (f) \). Then \( (f) \cap X = \emptyset \) which implies that \( f \) is constant. \( \square \)

1.3. Relationship to closed orbit. Recall that there’s an isomorphism \( L : X^*(T) \sim \text{Pic} G/B \) given by
\[
L(\lambda) := (G \times k_{-\lambda})/B.
\]
Define \( L_Y : X^*(T) \leftrightarrow \text{Pic} Y \) by
\[
L_Y(\lambda) := L(-w_0\lambda) \boxtimes L(\lambda)
\]

**Proposition.** Restriction \( \text{Pic} Y \leftarrow \text{Pic} X \) to \( Y \) is an injection onto \( X^*(T) \).

**Proof.** Let \( \pi \) denote \( G \rightarrow G_{\text{ad}} \subset X \). Since \( G \) is simply connected, we have \( \text{Pic} G = 0 \) and \( k[G]^* = k^* \). Mumford refers to Rosenlicht; see also Dolgachev. So \( \pi^* D_i = (f) \) with \( f \) unique up to scalar. So \( f \) is a \( B \times B \)-eigenvector.

Actually
\[
bf b^{-1} = \frac{\chi_i(b)}{w_0\chi_i(b')} f
\]
Now \( \mathcal{O}_X(D_i) \) has a linear \( G \) action (actually unique, but not sure we need this), and an equivariant trivialization
\[
\phi : \pi^* \mathcal{O}_X(D_i) \sim \mathcal{O}_G.
\]
As Dolgachev explains, there’s an exact sequence \( (G = G \times G) \)
\[
\text{Pic}^G(X) \rightarrow \text{Pic} X \xrightarrow{\delta} \text{Pic} G;
\]
the map \( \delta \) depends on the choice of an \( x \in X \), giving rise to
\[
G \xrightarrow{\iota} G \times X \xrightarrow{\alpha} X,
\]
in terms of which it is given by
\[
L \mapsto i^* \left( \frac{p^* L}{\alpha^* L} \right).
\]
Let \( \tau \in \Gamma(X, \mathcal{O}(D_i)) \) denote the canonical section. Then \( \phi(\tau) = f \) modulo scalars. So \( \tau \) is a \( B \times B \)-eigenvector of weight \( (\lambda_1 - w_0 \lambda) \). It follows that
\[
\mathcal{O}_X(D_i)|_Y = L_Y(\chi_i).
\]
The proposition follows. \( \square \)
2. **Line bundles and intersection multiplicities for** $SL(n + 1)$, 
\[
\sigma(A) = A^{-1}.
\]

2.1. **Setup.** Fix a dominant regular weight $\lambda$. We have a special isomorphism $V^*_\lambda \to V^\sigma_\lambda$, and a corresponding element $h \in V_{2\lambda} \subset V^\otimes_\lambda$. We let $X = G[h] \subset \mathbb{P}^V_\lambda$. We have a fibration of each orbit closure over a $G/P$. In particular, here $Y = G/B$. We have $\sigma(\alpha) = -\alpha$ so the subscript 1 occurring in dC-P goes away, and $l = n$. Analysis of a cell decomposition (with only even dimensional cells, maybe) shows that 
\[
\text{Pic } X = \mathbb{Z}^n.
\]
We have maps 
\[
G/B \hookrightarrow X \hookrightarrow \prod \mathbb{P}^V_{2\omega_i} \to \mathbb{P}^V_{2\omega_i}
\]
inducing 
\[
2\omega_i \hookrightarrow \mathcal{O}(1)
\]
on Pic. So we can identify Pic with twice the weight lattice. Under this identification $[X_i] = 2\alpha_i$. Since $\langle \omega_i, \alpha_j \rangle = 0$ for $i \neq j$, if for each $i$, $\lambda_i$ is either $2\omega_i$ or $2\alpha_i$, then $\{\lambda_1, \lambda_n\}$ is a basis of Pic$_\mathbb{Q}$.

2.2. **The algorithm.** We wish to compute an intersection multiplicity 
\[
(2\omega_1)^{h_1} \cdots (2\omega_n)^{h_n},
\]
\[
\sum h_i = \dim X.
\]
Consider more generally an intersection of the form 
\[
M = [X_1]^{g_1} \cdots [X_n]^{g_n} (2\omega_1)^{h_1} \cdots (2\omega_n)^{h_n}
\]
with $\sum g_i + h_i = \dim X$ and $g_i \leq 1$. If there exists an $i$ such that $g_i = 0$ but $h_i \neq 0$, we can write $M$ as a rational linear combination of $M$’s in which $g_i$ is increased by one and $h_i$ is decreased by one. In this way we obtain two types of $M$.

Type 1. There are still $i$’s for which $g_i = 0$. Then each $(2\omega_i)$ with nonzero exponent comes from the cohomology of $G/P_I$ ($I = \{i \mid g_i = 1\}$) which has lower dimension than $X_I$. So the intersection is zero.

Type 2. $X_I = Y = G/B$ is the closed orbit. Then $M$ is the intersection $(2\omega_1)^{h_1} \cdots (2\omega_n)^{h_n}$ on $G/B$ (which is computable by Borel: Sur la cohomologie des espaces fibrés principaux...).

3. **Counting quadrics**

3.1. Let $X$ denote the space of nondegenerate quadrics in $\mathbb{P}^3$, and let $Q_0$ denote the quadric 
\[
X_1^2 + X_2^2 + X_3^2 + X_4^2 = 0
\]
Then $SL(4)$ acts transitively with stabilizer at $Q_0$ 
\[
= \{A \mid A^t A \in k^*\} = N(SO(4)).
\]
Let $X$ be the wonderful compactification. Fix a flag 
\[
0 = \pi_0 \subset \pi_1 \subset \pi_2 \subset \mathbb{P}^3
\]
Let $E_i$ denote the closure of the divisor of quadrics tangent to $\pi_i$, and $E$ the closure of the divisor of quadrics tangent to $Q_0$. Then de Concini – Procesi prove that

- $E_i|_Y = L_Y(2\omega_i)$
- $E \sim 2(E_1 + E_2 + E_3)$

We can then write $E^9 = 2^9(E_1 + E_2 + E_3)$ as a sum of monomials in the $2\omega_i$, and follow the algorithm. The resulting number is equal to the number of points common to generic translates $g_iE_i$ inside the open orbit. This is in Kleiman: The transversality of a general translate.