Divisors and their intersections on wonderful compactifications *in* Essen

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Introduction

Let G be a semisimple and simply connected algebraic group over k, σ an automorphism of order 2, $H = G^\sigma$. The main point of this talk is to explain the following yoga: *If you want to understand line bundles and intersection multiplicities on $G/N(H)$, you should wonderfully compactify and go to the closed orbit.*

Section 1 will be about line bundles in the case $G \times G$, $\sigma(g, g') = (g', g)$.

Section 2) Line bundles and intersection multiplicities for the case SL_n, $\sigma(A) = {}^tA^{-1}$. This is harder, and we give fewer details.

Section 3) application to counting quadrics, a very superficial sketch.

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1.1. **Setup / review / notation.** Fix a Borel subgroup B.

Let $\Delta \subset X^*(T)$ be the set of roots (= nonzero weights for the action of T on $\mathfrak{g}$.

Let Δ^+ denote the set of positive roots (those occurring in $\mathfrak{n} = \operatorname{Lie} U$, U the unipotent radical of B).

Let $\{\alpha_1^\vee, , \alpha_l^\vee\} \subset X_*(T)$ denote the simple coroots,

$\{\chi_1, , \chi_l\} \subset X^*(T)$ the dual basis of "fundamental weights".

Fix a *regular, dominant* weight λ. Recall that this means that $\lambda = \sum n_i \chi_i$ with $n_i \neq 0$ for all i.

Let $M = M(\lambda)$ the associated simple representation of highest weight λ.

Consider the action of $G \times G$ on $\operatorname{End} M$ by $(g, g')(\phi) = g\phi g'^{-1}$.

Let h denote the identity element,

$[h]$ its image in $\mathbb{P}^\vee \operatorname{End} M = \operatorname{Proj} S^\bullet(\operatorname{End} M)^\vee =$ lines!

Let $X = \overline{(G \times G)[h]}$.

Let $h = \sum h_\mu$ be the decomposition of h for the action of $1 \times T$. Then actually

$$h = m_\lambda^* \otimes m_\lambda + \sum_{\mu < \lambda} h_\mu$$

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for $m_\lambda \in M_\lambda$, $m^*_\lambda \in M^*_{-\lambda}$, with $\langle m^*_\lambda, m_\lambda\rangle = 1$, meaning that λ has multiplicity one and the other μ's are smaller than λ.

Let $\underline{X}$ denote the open subset cut out by $m_\lambda \otimes m^*_\lambda \in (\operatorname{End} M)^*$. If we use $(-\alpha_1, , -\alpha_l)$ to embed T_{ad} in $\mathbb{A}^l$ then the map

$$T \to (T \times T)[h]$$

given by

$$t \mapsto (1,t)[h] = \lambda(t) \sum_{n_1,,n_l \in \mathbb{N}} \alpha_1(t^{-n_1}) \cdots \alpha_l(t^{-n_l}) h_{\lambda - \sum n_i \alpha_i}$$

extends to a closed immersion $\mathbb{A}^l \to \underline{X}$, which extends to an isomorphism

$$U \times \mathbb{A}^l \times U^- \xrightarrow{\sim} \underline{X}\,.$$

In X_0 we have the boundary divisor

$$U \times \mathbb{A}^l \times U^- \setminus U \times \mathbb{G}_m^l \times U^- = \underline{X}_1 + + \underline{X}_l$$

with closure

$$X \setminus (G \times G)[h] = X_1 + + X_l\,.$$

Then the closed orbit is given by

$$Y = \bigcap X_i = G/B \times G/B\,.$$

1.2. **Divisors coming from Bruhat decomp.** Let $W = N(T)/T$, denote by s_i the elt corresp to α_i, and by w_0 the longest element which interchanges Δ^+ with Δ^-.

Let $D_i = \overline{B_{\mathrm{ad}} s_i w_0 B_{\mathrm{ad}}}$.

Then

(1) Each D_i is a divisor. Let's call these the *Bruhat divisors.*
(2) $X \setminus \underline{X} = D_1 w_0 + + D_l w_0$
(3) $\operatorname{Pic} X$ is freely generated by the D_i.

Proof. (1) Recall that G_{ad} is stratified by subsets $B_{\mathrm{ad}} w B_{\mathrm{ad}}$, $w \in W$, and that there are isomorphisms

$$\mathbb{G}_a^{\Delta^+ \cap w(\Delta^-)} \times B_{\mathrm{ad}} \xrightarrow{\sim} B_{\mathrm{ad}} w B_{\mathrm{ad}}$$

So $\overline{B_{\mathrm{ad}} w B_{\mathrm{ad}}}$ is a divisor precisely when $w = s_i w_0$.

(2) We've seen that every orbit meets $\underline{X}$, so

$$\overline{G_{\mathrm{ad}} \setminus \underline{X}} = X \setminus \underline{X}$$

On the other hand, we have

$$\begin{aligned} G_{\mathrm{ad}} \cap \underline{X} &= U T_{\mathrm{ad}} U^- \\ &= B_{\mathrm{ad}} B^-_{\mathrm{ad}} \\ &= B_{\mathrm{ad}} w_0 B_{\mathrm{ad}} w_0 \end{aligned}$$

is the translate of the open Bruhat cell by w_0. So $G_{\mathrm{ad}} \setminus \underline{X}$ is a sum of w_0-translates of Bruhat divisors.

(3) Since $\operatorname{Pic}\underline{X} = 0$, $\mathbb{Z}\langle D_1, , D_l\rangle \twoheadrightarrow \operatorname{Pic} X$. Suppose $\sum a_i D_i = (f)$. Then $(f) \cap \underline{X} = \emptyset$ which implies that f is constant. □

1.3. **Relationship to closed orbit.** Recall that there's an isomorphism $L : X^*(T) \xrightarrow{\sim} \operatorname{Pic} G/B$ given by

$$L(\lambda) := (G \times k_{-\lambda})/B\,.$$

Define $L_Y : X^*(T) \hookrightarrow \operatorname{Pic} Y$ by

$$L_Y(\lambda) := L(-w_0\lambda) \boxtimes L(\lambda)$$

Proposition. Restriction $\operatorname{Pic} Y \leftarrow \operatorname{Pic} X$ to Y is an injection onto $X^*(T)$.

Proof. Let π denote $G \to G_{\mathrm{ad}} \subset X$. Since G is simply connected, we have $\operatorname{Pic} G = 0$ and $k[G]^* = k^*$. Mumford refers to Rosenlicht; see also Dolgachev. So $\pi^* D_i = (f)$ with f unique up to scalar. So f is a $B \times B$-eigenvector.

Actually

$$bfb'^{-1} = \frac{\chi_i(b)}{w_0\chi_i(b')} f$$

Now $\mathcal{O}_X(D_i)$ has a linear G action (actually unique, but not sure we need this), and an equivariant trivialization

$$\phi : \pi^*\mathcal{O}_X(D_i) \xrightarrow{\sim} \mathcal{O}_G\,.$$

As Dolgachev explains, there's an exact sequence ($G = G \times G$)

$$\operatorname{Pic}^G(X) \to \operatorname{Pic} X \xrightarrow{\delta} \operatorname{Pic} G\,;$$

the map δ depends on the choice of an $x \in X$, giving rise to

$$G \xrightarrow{i} G \times X \underset{p}{\overset{\alpha}{\rightrightarrows}} X\,,$$

in terms of which it is given by

$$L \mapsto i^*\left(\frac{p^*L}{\alpha^*L}\right)\,.$$

Let $\tau \in \Gamma(X, \mathcal{O}(D_i))$ denote the canonical section. Then $\phi(\tau) = f$ modulo scalars. So τ is a $B \times B$-eigenvector of weight $(\lambda, -w_0\lambda)$. It follows that

$$\mathcal{O}_X(D_i)|_Y = L_Y(\chi_i)\,.$$

The proposition follows. □

2. Line bundles and intersection multiplicities for $SL(n+1)$, $\sigma(A) = {}^t A^{-1}$.

2.1. **Setup.** Fix a dominant regular weight λ. We have a special isomorphism $V_\lambda^* \to V_\lambda^\sigma$, and a corresponding element $h \in V_{2\lambda} \subset V_\lambda^{\otimes 2}$. We let $X = \overline{G[h]} \subset \mathbb{P}^\vee V_{2\lambda}$. We have a fibration of each orbit closure over a G/P. In particular, here $Y = G/B$. We have $\sigma(\alpha) = -\alpha$ so the subscript 1 occurring in dC-P goes away, and $l = n$. Analysis of a cell decomposition (with only even dimensional cells, maybe) shows that

$$\operatorname{Pic} X = \mathbb{Z}^n \,.$$

We have maps

$$G/B \hookrightarrow X \hookrightarrow \prod \mathbb{P}^\vee V_{2\omega_i} \twoheadrightarrow \mathbb{P}^\vee V_{2\omega_i}$$

inducing

$$2\omega_i \hookleftarrow \mathcal{O}(1)$$

on Pic. So we can identify Pic with twice the weight lattice. Under this identification $[X_i] = 2\alpha_i$. Since $(\omega_i, \alpha_j) = 0$ for $i \neq j$, if for each i, λ_i is either $2\omega_i$ or $2\alpha_i$, then $\{\lambda_1, , \lambda_n\}$ is a basis of $\operatorname{Pic}_{\mathbb{Q}}$.

2.2. **The algorithm.** We wish to compute an intersection multiplicity

$$(2\omega_1)^{h_1} \cdots (2\omega_n)^{h_n} \,,$$

$\sum h_i = \dim X$. Consider more generally an intersection of the form

$$M = [X_1]^{g_1} \cdots [X_n]^{g_n} (2\omega_1)^{h_1} \cdots (2\omega_n)^{h_n}$$

with $\sum g_i + h_i = \dim X$ and $g_i \leq 1$. If there exists an i such that $g_i = 0$ but $h_i \neq 0$, we can write M as a rational linear combination of M's in which g_i is increased by one and h_i is decreased by one. In this way we obtain two types of M.

Type 1. There are still i's for which $g_i = 0$. Then each $(2\omega_i)$ with nonzero exponent comes from the cohomology of G/P_I ($I = \{ i \mid g_i = 1 \}$) which has lower dimension than X_I. So the intersection is zero.

Type 2. $X_I = Y = G/B$ is the closed orbit. Then $M =$ the intersection $(2\omega_1)^{h_1} \cdots (2\omega_n)^{h_n}$ on G/B (which is computable by Borel: Sur la cohomologie des espaces fibrés principaux...).

3. Counting quadrics

3.1. Let $\underline{X}$ denote the space of nondegenerate quadrics in $\mathbb{P}^3$, and let Q_0 denote the quadric

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = 0$$

Then SL(4) acts transitively with stabilizer at Q_0

$$= \{ A \mid A^t A \in k^* \} = N(\mathrm{SO}(4)).$$

Let X be the wonderful compactification. Fix a flag

$$0 = \pi_0 \subset \pi_1 \subset \pi_2 \subset \mathbb{P}^3$$

Let E_i denote the closure of the divisor of quadrics tangent to π_i, and E the closure of the divisor of quadrics tangent to Q_0. Then de Concini – Procesi prove that

- $E_i|_Y = L_Y(2\omega_i)$
- $E \sim 2(E_1 + E_2 + E_3)$

We can then write $E^9 = 2^9(E_1 + E_2 + E_3)$ as a sum of monomials in the $2\omega_i$, and follow the algorithm. The resulting number is equal to the number of points common to generic translates $g_i E_i$ inside the open orbit. This is in Kleiman: The transversality of a general translate.

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