Bloch's formula and the Gersten resolution *at the* baby seminar *in* Essen

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1. Theorem

1.1. Let X be regular of finite type over a field, let $\mathcal{K}_n$ denote the Zariski sheaf associated to $U \mapsto K_n(U)$, and let X^p denote the set of points of codimension p. Then there's a flasque resolution

$$0 \to \mathcal{K}_n \to \coprod_{x \in X^0} (i_x)_* K_n(x) \to \coprod_{x \in X^1} (i_x)_* K_{n-1}(x) \to \cdots \to \coprod_{x \in X^n} (i_x)_* K_0(x)$$

and an isomorphism

$$\mathrm{H}^n(X, \mathcal{K}_n) = \mathrm{CH}^n(X)\,.$$

2. The method of exact couples

2.1. Let $\mathcal{A}$ be an abelian category. Given an exact triangle (= long exact sequence)

$$\begin{array}{ccc} D_1 & \xrightarrow{b_1} & D_1 \\ {\scriptstyle a_1}\nwarrow & & \swarrow{\scriptstyle c_1} \\ & E_1 & \end{array}$$

we define a new exact triangle

$$\begin{array}{ccc} D_2 & \xrightarrow{b_2} & D_2 \\ {\scriptstyle a_2}\nwarrow & & \swarrow{\scriptstyle c_2} \\ & E_2 & \end{array}$$

as follows. Let $d_1 = c_1 a_1$. Let $Z_1 = \ker d_1$, $B_1 = \operatorname{im} d_1$, $E_1 = Z_1/B_1$. Let D_2 denote the canonically isomorphic objects $\operatorname{cok} a_1 = \operatorname{im} b_1 = \ker c_1$. Then

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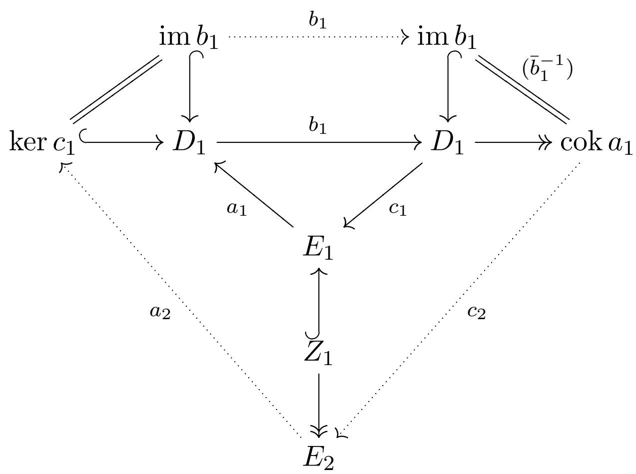
a_1, b_1, c_1 induce maps a_2, b_2, c_2 as in the following diagram.



In this way we obtain a filtration

$$E_1 = Z_1 \supset Z_2 \supset \cdots \supset B_2 \supset B_1 = 0.$$

Let

$$Z_\infty := \bigcap Z_i$$
$$B_\infty := \bigcap B_i$$
$$E_\infty := Z_\infty / B_\infty\,.$$

2.2. Sometimes exact couples arise from a family of long exact sequences. We number the terms of the p^{th} exact sequence like so.

$$\cdots \to A^{p+1,q-1} \xrightarrow{f_{p,q}} A^{p,q} \xrightarrow{g_{p,q}} E^{p,q} \xrightarrow{h_{p,q}} A^{p+1,q} \to \cdots\,.$$

We set $D_1 = \bigoplus A^{p,q}$, $E_1 = \bigoplus E^{p,q}$, which we endow, moreover, with the obvious bigrading, and we let $a_1 = h$, $b_1 = f$, $c_1 = g$. Since the construction of E_n, d_n preserves the grading, we obtain maps

$$d_n^{p,q} : E_n^{p,q} \to E_n^{p+n,q-n+1}$$

forming a spectral sequence.

2.3. We define decreasingly filtered objects A^n by

$$A^{p+q} := \varinjlim \big(\cdots \to A^{p,q} \xrightarrow{f} A^{p-1,q+1} \xrightarrow{f} A^{p-2,q+2} \to \cdots \big)$$
$$F^p A^{p+q} := \text{ image of } A^{p,q}\,.$$

We assume from now on that the direct system above is zero eventually to the left and constant eventually to the right. Then for n sufficiently large there's an isomorphism

$$\operatorname{gr}^p A^{p+q} \xrightarrow{\sim} E_n^{p,q}\,.$$

The map is constructed as follows. Recall that $D_m^{s,t}$ is a subobject of $D_1^{s,t}$ which was defined to be $A^{s,t}$. So for n large, $D_n^{p-n+1,q+n-1}$ is a subobject of $A^{p,q}$. In fact, for such n, this is precisely the p^{th} filtered piece. The map is then given by c_n.

3. BGQ SPECTRAL SEQUENCE

3.1. Suppose X Noetherian. Let $\mathcal{M}^p$ denote the category of coherent sheaves supported in codimension $\geq p$, $\mathcal{A}(R)$ the category of modules of finite length. The functor

$$\psi : \mathcal{M}^p \to \coprod_{x \in X^p} \mathcal{A}(\mathcal{O}_{X,x}) \qquad \text{given by} \qquad \psi(E) = (E_x)_{x \in X^p}$$

factors through an isomorphism

$$\mathcal{M}^p/\mathcal{M}^{p+1} \xrightarrow{\sim} \coprod_{x \in X^p} \mathcal{A}(\mathcal{O}_{X,x}) \,.$$

Let's content ourselves with a construction of the factorization. Recall that for $E, F \in \mathcal{M}^p$,

$$\operatorname{Hom}_{\mathcal{M}^p/\mathcal{M}^{p+1}}(E, F) = \varinjlim \operatorname{Hom}(E', F'')$$

in which the limit ranges over short exact sequences

$$0 \to E' \to E \to E'' \to 0 \qquad\qquad 0 \to F' \to F \to F'' \to 0$$

with $E'', F' \in \mathcal{M}^{p+1}$. The alleged factorization

$$\begin{array}{ccc} \operatorname{Hom}(E,F) & \longrightarrow & \bigoplus_{x \in X^p} \operatorname{Hom}(E_x, F_x) \\ \big\downarrow & \overset{\cong}{\nearrow} & \\ \varinjlim \operatorname{Hom}(E', F'') & & \end{array}$$

is defined as follows.

$$\begin{array}{ccc} E' & \longrightarrow & F'' \\ \big\downarrow & & \big\uparrow \\ E & & F \end{array} \qquad \mapsto \qquad \begin{array}{ccc} E'_x & \longrightarrow & F''_x \\ \cong\big\downarrow & & \big\uparrow\cong \\ E_x & & F_x \end{array}$$

By devissage, $K_q\mathcal{A}(\mathcal{O}_{X,x})) = G_q(x)$, so we obtain localization sequences like so.

$$K_i\mathcal{M}^{p+1} \to K_i\mathcal{M}^p \to \coprod_{x \in X^p} G_i(x) \xrightarrow{\partial} K_{i-1}\mathcal{M}^{p+1}$$

Setting

$$A^{p,q} = K_{-p-q}\mathcal{M}^p$$
$$E_{p,q} = \bigoplus_{x \in X^p} G_{-p-q}(x)$$

we obtain a spectral sequence. Its first page is defined like so,

$$\begin{array}{ccccc} \coprod_{x\in X^p} G_i(x) & \longrightarrow & K_{i-1}\mathcal{M}^{p+1} & \xrightarrow{\partial} & \coprod_{x\in X^{p+1}} G_{i-1}(x) \\ \| & & & & \| \\ E_1^{p,-p-i} & & \xrightarrow[d_1]{\hspace{6em}} & & E_1^{p+1,-p-i} \end{array}$$

and looks like so.

$$\begin{array}{l} \bigoplus_0 G_0 \\ \bigoplus_0 G_1 \longrightarrow \bigoplus_1 G_0 \\ \bigoplus_0 G_2 \longrightarrow \bigoplus_1 G_1 \longrightarrow \bigoplus_2 G_0 \end{array}$$

The abutment is given by

$$A^{p+q} = \varinjlim \left(\cdots \to K_{-p-q}\mathcal{M}^p \to K_{-p-q}\mathcal{M}^{p-1} \to \cdots \right) = G_{-p-q}(X)$$

with filtration $F^p A^{p+q}$ = image of $K_{-p-q}\mathcal{M}^p$.

4. The Gersten resolution

4.1. Theorem. Suppose X is of finite type over a field k, and let $x \in X$ be a regular point. Then for each i, the map $K_i\mathcal{M}^{p+1}(\mathcal{O}_{X,x}) \to K_i\mathcal{M}^p(\mathcal{O}_{X,x})$ is zero.

4.2. Here's a rough sketch based on Srinivas, which is limited to the case that the ground field k is infinite. We reduce to showing: suppose X smooth over k, $Y \hookrightarrow X$ a closed immersion cut out by a single nonzero divisor. Then locally the map

$$K_i\mathcal{M}^p(Y) \to K_i\mathcal{M}^p(X)$$

is zero. We arrange the following situation.

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & \mathbb{A}^n \end{array} \qquad\qquad \begin{array}{ccc} 0 & & \\ \downarrow & & \\ J' & & \\ \downarrow & & \\ A' & \longleftarrow & A \\ \downarrow\uparrow & & \uparrow \\ B' & \longleftarrow & B \\ \downarrow & & \\ 0 & & \end{array}$$

The square is cartesian, g is finite, and f' is smooth. Now to any $M \in \mathcal{M}^p(Y)$ we can associated the short exact sequence

$$0 \to J' \otimes_{B'} M \to A' \otimes_{B'} M \to M \to 0\,.$$

Then

1. (1) I principal implies that $J' \otimes M \cong A' \otimes M$,
2. (2) f' flat implies that $f'^* M = A' \otimes M \cong J' \otimes M$ is supported in codim $\geq p$, and
3. (3) g finite implies that g'_* preserves $\mathcal{M}^p$.

Conclusion: have short exact sequence of functors with the first two inducing the same maps on K groups.

4.3. Corollary. Suppose X regular local, essentially of finite type over k. Then the sequence

$$0 \to G_n(X) \to \coprod_{x \in X^0} G_n(x) \to \coprod_{x \in X^1} G_{n-1}(x) \to \cdots$$

is exact.

Proof. We have short exact sequences like so.

$$0 \to K_i \mathcal{M}^p \to \coprod_{x \in X^p} G_i(x) \to K_{i-1}\mathcal{M}^{p+1} \to 0$$

□

4.4. Corollary. Let $\mathcal{G}$ denote the Zariski sheaf associated to $U \mapsto G_n(U)$. Then $E_2^{p,q} = H^p(X, \mathcal{G}_{-q})$.

5. Poof

5.1. CH^p is the cokernel of ord:

$$\coprod_{y \in X^{p-1}} k(y)^* \to \coprod_{x \in X^p} \mathbb{Z} \to \mathrm{CH}^p(X) \to 0.$$

Given $x \in Y = \overline{\{y\}}$, its (x, y)-component is the map $k(y)^* \to \mathbb{Z}$ define as follows. Write $a \in k(y)^*$ as $a = b/c$, $b, c \in \mathcal{O}_{Y,x}$. Then

$$\mathrm{ord}(a) = \mathrm{length}(\mathcal{O}_{Y,x}/(b)) - \mathrm{length}(\mathcal{O}_{Y,x}/(c))\,.$$

Claim. $\mathrm{ord} = d_1$

The problem is that d_1 involves the boundary map in a long exact sequence of homotopy groups of classifying spaces of complicated categories. We don't even attempt to analyze it directly. Instead, by various functorialities and compatibilities we reduce to: R a Noetherian local domain over k of dimension 1, with residue field k' and quotient field F. Then the diagram

$$\begin{array}{ccccc} G_1(R) & \longrightarrow & G_1(F) & \xrightarrow{\partial} & G_0(k') \\ \| & & \| & & \| \\ R^* & \longrightarrow & F^* & \xrightarrow[\mathrm{ord}]{} & \mathbb{Z} \end{array}$$

commutes. The first square is clear, and is really only there to assist us in proving that the second square commutes. Indeed, with its help, we settle

Case 1. $x \in R^*$. Then $\operatorname{ord} x = \operatorname{length}(R/(x)) = 0$, so we're ok.

Case 2. $x \notin R^*$. Consider the map to $\mathbb{A}^1$ defined by x: $R \leftarrow k[t]$. Then we have morphisms of open/closed complement situations and associated morphisms of localization sequences like so.

$$\begin{array}{ccccc} k' & \twoheadleftarrow & R & \hookrightarrow & F \\ \twoheaduparrow & & \| & & \| \\ R/x & \twoheadleftarrow & R & \hookrightarrow & F \\ \uparrow & & \| & & \uparrow \\ k & \twoheadleftarrow & k[t] & \longrightarrow & k[t,t^{-1}] \end{array} \qquad \begin{array}{ccccc} G_1(R) & \longrightarrow & G_1(F) & \longrightarrow & G_0(k') \\ \| & & \| & & \downarrow \cong \\ G_1(R) & \longrightarrow & G_1(F) & \longrightarrow & G_0(R/x) \\ \uparrow & & \uparrow & & \uparrow \\ G_1(k[t]) & \longrightarrow & G_1(k[t,t^{-1}]) & \longrightarrow & G_0(k) \end{array}$$

The isomorphism on the upper right is given by devissage, since R/x is Artinian. We see that $t \in G_1(k[t,t^{-1}])$ maps to $x \in F^*$, and that $1 \in G_0(k)$ maps to $\operatorname{ord} x = \operatorname{length} R/x \in G_0(k')$. So we're reduced to analyzing the boundary map

$$\partial : G_1(k[t,t^{-1}]) \to G_0(k)\,.$$

But in this case we were able to show in last week's talk that the map is surjective, which suffices.

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