The unipotent fundamental group is motivic in the Motives seminar in Essen

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0.1. Let X be smooth over a number field k. Let $a, b \in X(k)$. Deligne defines a prounipotent torsor object in systems of realizations ${}_bP_a^{\mathrm{real}}$. Let's recall briefly its Betti and de Rham realizations, on which we'll focus for this talk. For the Betti realization, we consider the category of unipotent local systems on X^B, and we let ${}_bP_a^{\mathrm{B}}$ be the torsor of Tannakian paths from the fiber functor associated to a to the fiber functor associated to b. We further endow ${}_bP_a^{\mathrm{B}}$ with a certain integral structure. For the de Rham realization, we consider the category of unipotent vector bundles with integrable connection. We let ${}_bP_a^{\mathrm{dR}}$ be the torsor of Tannakian paths. We further endow ${}_bP_a^{\mathrm{dR}}$ with a certain filtration, the *Hodge filtration*. The idea of endowing ${}_bP_a^{\mathrm{dR}}$ with such a filtration is due to Hain, who works in an analytic context. The general construction in an algebraic context is due to Wojtkowiak.

0.2. Theorem. Suppose X is mixed Tate. Then ${}_bP_a^{\mathrm{real}}$ is the realization of a prounipotent tosor object ${}_bP_a$ of the category of mixed Tate Voevodsky motives.

0.3. The Betti construction, as presented by Wojtkowiak, relies on Chen's theory of iterated integrals. Wojtkowiak shows how to retrieve the n-nilpotent functions on the Betti path torsor

$$\mathbb{Q}[{}_bP_a]/I^{n+1} \to \mathbb{Q}$$

from the diagram of dga's

$$C^*(X, \mathbb{Q}) \rightrightarrows \mathbb{Q}$$

using the bar construction.

0.4. The de Rham construction, also due to Wojtkowiak, relies on work of Hain and Zucker. The de Rham path space ${}_bP_a^{\mathrm{dR}}$ may be retrieved from the diagram of dga's

$$\Omega^*(X) \rightrightarrows k$$

using the bar construction.

0.5. Goncharov notes the existence of a t-structure on the triangulated category of mixed Tate Voevodsky motives for a field which satisfied Beilinson-Soulé vanishing, and repeats Wojtkowiak's construction in Voevodsky's triangulated category.

Actually, Goncharov approaches the Betti realization thru an alternative construction due to Beilinson (c.f. Goncharov's unpublished manuscript *Multiple polylogarithms and mixed Tate motives*). Beilinson first retrieves the n-nilpotent functions from a certain constructible hypercohomology on X^n. This is our first main goal. We follow the exposition in Deligne-Goncharov.

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1. Beilinson's Constructible hypercohomology

1.1. Let X be a well-behaved topological space, and let $a, b \in X$. Let $\pi_1 = \pi_1(X, a)$. Given a π_1-module E, we let $\widetilde{E}$ denote the assoc local system. Define a divisor

$$Y = {}_bY_a\langle n\rangle = \sum Y_i$$

on X^n by

$$\begin{aligned} Y_0 &= \{b = t_1\} \\ Y_1 &= \{t_1 = t_2\} \\ &\vdots \\ Y_n &= \{t_n = a\}\,. \end{aligned}$$

We write Y_I for the intersection of components associated to $I \subset \{0, \dots, n\}$, and i_I its inclusion in X^n. Let

$$k_I = k_{Y_I} = i_{I*}\underline{k}\,.$$

Let $\mathcal{K} = {}_a\mathcal{K}_b\langle n\rangle$ denote the complex

$$\mathcal{K}^p = \bigoplus_{|I|=p} k_I$$

($p = 0, \dots, n-1$), with boundary maps given by the obvious restriction maps with appropriate signs. We replace b with a $*$ to denote the same construction with b variable:

$${}_*\mathcal{K}_a\langle n\rangle$$

is a constructible complex on $X \times X^n$.

1.2. Proposition. (Absolute version.) $H^i(X^n, {}_b\mathcal{K}_a\langle n\rangle) = 0$ for $i < n$, $= (\mathbb{Q}[{}_bP_a]/I^{n+1})^\vee$ for $i = n$.

1.3. Proposition. (Relative version.) $(R^ip_*)_*\mathcal{K}_a\langle n\rangle = 0$ for $i < n$, $= (\widetilde{\mathbb{Q}[\pi_1]/I^{n+1}})^\vee$ for $i = n$.

1.4. The relative and absolute versions are presumably equivalent. We don't go into this. We note only that

$$(\widetilde{\mathbb{Q}[\pi_1]/I^{n+1}})_b = \mathbb{Q}[{}_bP_a]/I^{n+1}$$

and that according to Deligne-Goncharov, a devissage argument shows that the pushforward of the relative version yields the absolute cohomology of the absolute version upon pullback to b:

$$((R^ip_*)_*\mathcal{K}_a\langle n\rangle)_b = H^i(X^n, {}_b\mathcal{K}_a\langle n\rangle)\,.$$

1.5. Assuming the equivalence of the absolute and rel. versions, and assuming the prop. holds for $n = 1$, we sketch the argument for $n = 2$. On the chalk board, which represents X^2, there now appears the vertical line Y_0 at $x = b$, the horizontal line Y_2 at $y = a$, and the diagonal line Y_1, forming a triangle. The intersection points (a, a), (b, a), (b, b) are also emphasized. We have ${}_b\mathcal{K}_a\langle 2\rangle =$

$$0 \to \mathbb{Q}_{X^2} \to \mathbb{Q}_{Y_0} \oplus \mathbb{Q}_{Y_1} \oplus \mathbb{Q}_{Y_2} \to \mathbb{Q}_{(b,b)} \oplus \mathbb{Q}_{(b,a)} \oplus \mathbb{Q}_{(a,a)} \to 0$$

in degrees 0, 1, and 2, ${}_*\mathcal{K}_a\langle 1\rangle =$

$$0 \to \mathbb{Q}_{X^2} \to \mathbb{Q}_{Y_1} \oplus \mathbb{Q}_{Y_2} \to 0$$

in degrees $0, 1$, and $(i_{b*})({}_b\mathcal{K}_a\langle 1\rangle) =$

$$0 \to \mathbb{Q}_{Y_0} \to \mathbb{Q}_{(b,a)} \oplus \mathbb{Q}_{(b,b)} \to 0$$

in degrees $0, 1$, whence a cone sequence

$${}_*\mathcal{K}_a\langle 1\rangle \to (i_{b*})({}_b\mathcal{K}_a\langle 1\rangle) \oplus (\mathbb{Q}_{(a,a)}[-1]) \to {}_b\mathcal{K}_a\langle 2\rangle[1] \to {}_*\mathcal{K}_a\langle 1\rangle\,,$$

from which

$$\begin{array}{ccccccc}
0 \longrightarrow & R^0q_*({}_b\mathcal{K}_a\langle 2\rangle) & \longrightarrow & R^0q_*({}_*\mathcal{K}_a\langle 1\rangle) & \longrightarrow & R^0q_*(i_{b*})({}_b\mathcal{K}_a\langle 1\rangle) & \\
 & & & \| & & \| & \\
 & & & 0 & & 0 & \\
\longrightarrow & R^1q_*({}_b\mathcal{K}_a\langle 2\rangle) & \longrightarrow & R^1q_*({}_*\mathcal{K}_a\langle 1\rangle) & \longrightarrow & R^1q_*(i_{b*})({}_b\mathcal{K}_a\langle 1\rangle) \oplus \mathbb{Q}_{(a,a)} & \\
 & & & \| & & \| & \\
 & & & \left(\widetilde{\mathbb{Q}[\pi_1]/I^2}\right)^\vee & \twoheadrightarrow & (\mathbb{Q}[{}_bP_a/I^2])_b^\vee & \\
\longrightarrow & R^2q_*({}_b\mathcal{K}_a\langle 2\rangle) & \longrightarrow & R^2q_*({}_*\mathcal{K}_a\langle 1\rangle) & \longrightarrow & R^2q_*(i_{b*})({}_b\mathcal{K}_a\langle 1\rangle) & \longrightarrow \cdots
\end{array}$$

upon pushing forward along the first projection q, from which

$$R^0q_*({}_b\mathcal{K}_a\langle 2\rangle) = 0\,, \tag{*0}$$

and

$$0 \to R^1q_*({}_b\mathcal{K}_a\langle 2\rangle) \to (\mathbb{Q}[\pi_1]/I^2)^{\sim\vee} \to (\mathbb{Q}[{}_bP_a]/I^2)_b^\vee \oplus \mathbb{Q}_{(a,a)} \to 0\,, \tag{*1}$$

and

$$0 \to R^2q_*({}_b\mathcal{K}_a\langle 2\rangle) \to R^2q_*({}_*\mathcal{K}_a\langle 1\rangle) \to H^2({}_b\mathcal{K}_a\langle 1\rangle)_b \to 0\,. \tag{*2}$$

Moreover, both surjections are actually injective on global sections. Moreover,

$$R^2q_*({}_*\mathcal{K}_a\langle 1\rangle)$$

is locally constant. Indeed, from the stupid filtration we get

$$H^2({}_b\mathcal{K}_a\langle 1\rangle) = H^2(X,\mathbb{Q})\,.$$

The relevant part of the Leray spectral sequence looks like so.

$$\begin{array}{cccc}
0 & H^1(X, R^1q_*({}_b\mathcal{K}_a\langle 2\rangle)) & & \\
 & & \searrow & \\
0 & 0 & 0 & 0
\end{array}$$

So

$$H^2(X^2, {}_b\mathcal{K}_a\langle 2\rangle) = H^1(X, R^1q_*({}_b\mathcal{K}_a\langle 2\rangle))\,,$$

so we should analyze the latter.

1.6. Going back to (*1), we see that there is a natural equivalence from the category of extensions

$$0 \to R^1q_*({}_b\mathcal{K}_a\langle 2\rangle) \to E \to \mathbb{Q}_X \to 0$$

to the category of extensions

$$0 \to (\mathbb{Q}[\pi_1]/I^2)^{\sim\vee} \to E \to \mathbb{Q}_X \to 0$$

endowed with a splitting at b, and with a splitting of

$$0 \to \mathbb{Q}_X \to E^\epsilon \to \mathbb{Q}_X \to 0$$

at a. Here E^ϵ denotes the pushout of E along the counit

$$\epsilon : \mathbb{Q}[\pi_1] \to \mathbb{Q}\,.$$

1.7. It's convenient at this point to switch to the language of π_1-reps. Let G be a group and let P be a torsor. Then there's a P-twist G_P of G: its elements are pairs (p, g) (think pgp^{-1}) subject to the equivalence relation

$$(p, g) \sim (p', g') \qquad \text{if} \qquad gp^{-1}p' = p^{-1}p'g' \,.$$

Multiplication is determined by

$$(p, g)(p, g') = (p, gg') \,.$$

Given a representation E, there's a P-twist E_P of E, a representation of G_P. Its elements are pairs (p, e) subject to

$$(p, e) \sim (p', e') \qquad \text{if} \qquad p^{-1}p'e' = e \,,$$

and G_P-action determined by the formula

$$(p, g)(p, e) = (p, gg')$$

We apply this to our situation with $G = \pi_1/I^3$ and $P = {}_bP_a/I^3$, using a subscript b for to save space.

1.8. In this language, we are to construct a $\mathbb{Q}$-valued pairing between $(\pi_1/I^3)_b$ and the group of connected components for the category whose objects consist of (1) an extension of π_1-reps

$$0 \to \mathbb{Q} \to E \to \mathbb{Q}[\pi_1]/I^2 \to 0 \,,$$

(2) a splitting

$$\begin{array}{ccc} E' & \overset{\alpha}{\leftrightarrows} & \mathbb{Q} \\ \downarrow & & \downarrow \\ E & \longrightarrow & \mathbb{Q}[\pi_1]/I^2 \end{array}$$

and (3) a splitting

$$0 \longrightarrow \mathbb{Q} \overset{\beta}{\leftrightarrows} E_b \longrightarrow (\mathbb{Q}[\pi_1]/I^2)_b \longrightarrow 0 \,.$$

To do so, we note that the vector $\alpha(1)$ determines a map

$$\mathbb{Q}[\pi_1] \to E$$

which factors modulo I^3. Twisting by ${}_bP_a$, we obtain a map

$$(\mathbb{Q}[\pi_1]/I^3)_b \to E_b \,,$$

which we may then compose with β.

1.9. If you were paying attention, you'd have noticed that this argument only works for $b \neq a$.

2. The motivic construction

2.1. The cosimplicial model of the path space, denoted ${}_bX_a$, is defined by

$${}_bX_a{}^n = X^n$$

with boundary maps

$$\partial_i : X^{n-1} \to X^n$$

given by

$$\begin{aligned} \partial_0(t_1, \ldots, t_{n-1}) &= (b, t_1, \ldots, t_{n-1}) \,, \\ \partial_i(t_1, \ldots, t_{n-1}) &= (t_1, \ldots, t_i, t_i, \ldots, t_{n-1}) \,, \\ \partial_n(t_1, \ldots, t_{n-1}) &= (t_1, \ldots, t_{n-1}, a) \,. \end{aligned}$$

For any n, $\infty \geq n \geq 1$, the totalization

$$\hom(\Delta, {}_bX_a{}^{\leq n})$$

is homotopy equivalent to the space of paths

$$a \to b\,,$$

so

$$\pi_0 \hom(\Delta, {}_bX_a{}^{\leq n}) = {}_bP_a$$

and

$$H^0(\hom(\Delta, {}_bX_a{}^{\leq n}), \mathbb{Q}) = \mathbb{Q}[{}_bP_a]^\vee\,.$$

2.2. Let $S^*_\bullet$ denote the simplical object of the category of $\mathbb{Q}$-vector spaces given by

$$S^*_m = C^*(X^m, \mathbb{Q})\,.$$

There are various ways of attaching, for each n, an n-truncated complex to $S^*_\bullet$, each with its own advantages. Wojtkowiak simply takes alternating sums of boundary maps, and then takes the total complex. To truncate, he takes the stupid truncation (always in the simplicial variable!). Let's denote the resulting complex (resp. truncated complex) by S^*_* (resp. $S^*_{*\leq n}$). With this version it's relatively easy to see that as a consequence of the Künneth formula we have

$$S^*_* = \overline{B}(S^*_1 \rightrightarrows S^*_0)$$

Closely related is the complex

$$\sigma_{\leq n} N S^*_\bullet$$

considered by Deligne-Goncharov. More different is the complex

$$C_*(\Delta_n, S^*_\bullet)$$

with

$$C_p(\Delta_n, S^*_\bullet) = \bigoplus_{\tau \subset \Delta_n,\ |\tau| = p+1} S^*_p\,;$$

it has the advantage that it's relatively easy to see that

$$H^0 C_*(\Delta_n, S^*_\bullet) = H^n(X^n, {}_bK_a\langle n\rangle)\,.$$

2.3. Let X be smooth over a field satisfying Beilinson-Soulé vanishing. Suppose that X is mixed Tate. We regard ${}_bX_a$ as a cosimplicial object of $\mathrm{SmCor}(k)$. The associated truncated complex (whichever variant) gives rise to an object of the triangulated category of mixed Tate Voevodsky motives. We may then take H^0 with respect to the canonical t-structure.

Definition. We set

$${}_bA_a = \lim_{\to} (H^0 \sigma_{\leq n} N\, {}_bX_a)^\vee$$

and

$${}_bP_a = \operatorname{Spec} {}_bA_a\,.$$

3. De Rham realization

3.1. It remains to explain the relationship with the Tannakian fundamental group for unipotent connections. This will have to be very brief. Wojtkowiak constructs an affine scheme

$$ {}_*P_a^W \to X $$

with the structure of a torsor over

$$ \pi_1^W := \operatorname{Spec} H^0 \overline{B}(\Omega(X) \rightrightarrows k) $$

equipped with a connection as well as a trivialization at a. Given a unipotent representation

$$ \rho : \pi_1^W \to \mathrm{GL}(V) $$

we form the product

$$ E_\rho := {}_*P_a^W \overset{\rho}{\times} V \, . $$

The connection on ${}_*P_a^W$ endows E_ρ with the structure of a unipotent connection.

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