BEILINSON'S CONJECTURES ON VALUES OF L-FUNCTIONS IN THE MOTIVE SEMINAR IN ESSEN NOVEMBER 2014

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Abstract. We state Beilinson's conjectures on values of L-Functions. Our main reference is Nekovar's notes [Nek]. I hope in particular that section 6 about the class number formula will serve as a useful complement to Nekovar's notes. Please note that the apparent trichotomy in Beilinson's formulation and in the formulation below is explained, for instance, in Flach [Fla].

1. Convergence, analytic continuation, and functional equation

1.1. We follow Nekovar's notes in the 1994 Motives volumes. Following Nekovar we make some simplifying assumptions. In particular, we consider an X which is smooth proper over $\mathbb{Q}$ and we work with the motive $M = h^i(X)(n)$ with coefficients in $\mathbb{Q}$, which we can think of as being a system of realizations. We then have

$$M_l = H^i(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)(n).$$

We define

$$L_p(M,s) = \det(1 - p^{-s} Fr_p \mid M_l^{I_p})^{-1},$$

conjecturally independent of $l \neq p$. We define

$$L(M,s) = \prod_p L_p(M,s)$$

where p ranges over the (finite) primes of $\mathbb{Q}$.

1.2. We have

$$M_B = H^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(n)).$$

Here $\mathbb{Q}(1)_B$ is the Hodge structure $(2\pi i)\mathbb{Q}$ with Hodge type $(-1,-1)$ and involution $\phi_\infty = -1$. Together with the action of complex conjugation ϕ_∞, M_B is a Hodge structure with involution. In particular, there's a Hodge decomposition

$$M_B \otimes \mathbb{Q} = \bigoplus_{p+q=w} H^{p,q}$$

where $w = i - 2n$ and an involution ϕ_∞ on M_B such that $\phi_\infty \otimes 1$ sends $H^{p,q}$ to $H^{q,p}$. Let $h^{p,q}$ denote the dimension of $H^{p,q}$ and let $h^{p\pm}$ denote the $\pm(-1)^p$-eigenspace of ϕ_∞ acting on $H^{p,p}$. Let

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$
$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Then

$$L_\infty(M,s) = \prod_{p<w/2} \Gamma_{\mathbb{C}}(s-p)^{h^{p,w-p}} \cdot \Gamma_{\mathbb{R}}(s-w/2)^{h^{w/2+}} \cdot \Gamma_{\mathbb{R}}(s-w/2+1)^{h^{w/2-}}.$$

We let

$$\Lambda(M,s) = L_\infty(M,s)L(M,s).$$

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1.3. Some basic properties. For v finite of good reduction, L_v has poles of real part $w/2$ (by the Weil conjectures). For v finite of bad reduction, the purity conjecture for the monodromy filtration says that the poles have real parts

$$w/2, w/2-1/2, \ldots, w/2-i/2.$$

Also for all v we have

$$L_v(M(m), s) = L_v(M, s+m).$$

1.4. Conjecture. $L(M, s)$ is absolutely convergent for

$$Re(s) > w/2 + 1$$

(this part depends on the purity conjecture) and has a meromorphic continuation to the entire complex plane which satisfies

$$\Lambda(M, s) = \epsilon(M, s)\Lambda(w + 1 - s)$$

with

$$\epsilon(M, s) = ae^{bs}$$

for some $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$.

1.5. There are also precise conjectures about a and b, which is a long story in its own right; see Deninger's article in the Motives volumes and the references there.

2. Symmetry for orders and for leading terms

2.1. We have

$$M_{\mathrm{dR}} = H^i_{\mathrm{dR}}(X)(n),$$

a $\mathbb{Q}$-vector space with filtration

$$F^k M_{\mathrm{dR}} = H^i(X, \Omega^{\geq k+n}).$$

There is an isomorphism

$$I_\infty : M_B \otimes \mathbb{C} \xrightarrow{\cong} M_{\mathrm{dR}} \otimes \mathbb{C}.$$

Denote complex conjugation by c. Then under I_∞, $\phi_\infty \otimes c$ corresponds to $1 \otimes c$.

2.2. Example. Let $X = \mathbb{G}_m$, $i = 1$, $n = 0$. Then $h^1(X) = \mathbb{Q}(-1)$. We have M_B generated by the winding number W, M_{dR} generated by dz/z and

$$I_\infty(dz/z \otimes 1) = W \otimes 2\pi i.$$

2.3. Taking $\phi_\infty \otimes c$-invariants on the left and $1 \otimes c$-invariants on the right, we obtain an isomorphism

$$(M_B^+ \otimes \mathbb{R}) \oplus (M_B^- \otimes \mathbb{R}(1)).$$

So there's an induced map

$$\alpha_M : M_B^+ \otimes \mathbb{R} \to (M_{\mathrm{dR}}/F^0) \otimes \mathbb{R}.$$

2.4. Proposition. Suppose $w = i - 2n \leq -1$. Then
$$\dim_{\mathbb{R}}(\operatorname{Coker} \alpha_M) = \operatorname{ord}_{s=i+1-n} L(h^i(X), s) - \operatorname{ord}_{s=n} L(h^i(X), s).$$

Proof. By the functional equation
$$\begin{aligned} RHS &= \operatorname{ord}_n L_\infty(h^i(X)) - \operatorname{ord}_{i+1-n} L_\infty(h^i(X)) \\ &= \operatorname{ord}_0 L_\infty(M) - \operatorname{ord}_0 L_\infty(M^\vee(1)). \end{aligned}$$

$$L_\infty(M) = hol \cdot \prod_{p<w/2} \Gamma(s-p)^{h^{p,w-p}} \Gamma\left(\frac{s-w/2}{2}\right)^{h^{w/2+}} \Gamma\left(\frac{s-w/2+1}{2}\right)^{h^{w/2-}}$$
with hol holomorphic. So $\operatorname{ord}_0 L_\infty(M) = 0$. We compute the order of $M^\vee(1)$. We have
$$\begin{aligned} h^{p,q}(M^\vee(1)) &= h^{1-p,1-q}(M) \\ h^{w/2\pm}(M^\vee(1)) &= h^{w/2}(M). \end{aligned}$$
We plug in zero as a book-keeping device. We have
$$L_\infty(M^\vee(1), 0) = hol \cdot \prod_{p<q} \Gamma(-p)^{h^{1-p,1-q}} \cdot \Gamma(w/4 - 1/2)^{h^{w/2+}} \Gamma(w/4)^{h^{w/2-}}$$
so the order of pole is
$$\sum_{w/2<p<0} h^{p,q} + \dim H^{w/2,w/2-}$$
regardless of the residue class of $w \mod 4$. Attacking from the other direction, we have
$$\begin{aligned} \dim H/F^0 &= \sum_{p<0} h^{p,w-p} \\ &= \sum_{w/2<p<0} h^{p,w-p} + h^{w/2,w/2} + \sum_{-n\leq p<w/2} h^{p,w-p} \end{aligned}$$
and
$$\dim_{\mathbb{Q}} M_B^+ = \dim_{\mathbb{C}} H^+ = h^{w/2+} + \sum_{-n\leq p<w/2} h^{p,w-p}.$$
□

2.5. Remark. We have
$$\operatorname{Ker} \alpha_M \subset (M_B \otimes \mathbb{R}) \cap F^0(M_B \otimes \mathbb{C}) \subset (M_B)^{0,0}_{\mathbb{C}}$$
which equals 0 if $w \neq 0$.

2.6. We define two $\mathbb{Q}$-structures $D(M)$ and $B(M)$ on $\operatorname{Coker} \alpha_M$ under the assumption that $w \leq -1$. We saw that under this assumption, α_M is injective, so taking determinants, we obtain the $\mathbb{Q}$-structure $D(M)$ induced by the $\mathbb{Q}$-structures induced by those on the domain and target of α_M.

Proposition. Suppose $w \leq -1$. Then the maps

$$\begin{array}{ccc} (M^\vee(1)^+_B)^\vee \otimes \mathbb{R} & \twoheadrightarrow & \mathrm{Ker}(\alpha_{M^\vee(1)})^\vee \\ \| & & \\ M^-_B(-1) \otimes \mathbb{R} & & \vdots\ \cong \\ \downarrow I_\infty(M) & & \downarrow \\ M_{\mathrm{dR}} \otimes \mathbb{R} & \twoheadrightarrow & \mathrm{Coker}(\alpha_M) \end{array}$$

induce an isomorphism as shown.

Corollary. Assume $w \leq -1$. Then $\alpha_{M^\vee(1)}$ is surjective.

Proof. We have $\mathrm{Coker}(\alpha_{M^\vee(1)}) = \mathrm{Ker}(\alpha_M)^\vee = 0$. $\square$

Definition. By the corollary, $\mathrm{Ker}\, \alpha_{M^\vee(1)}$ inherits a $\mathbb{Q}$-structure from those on the domain and target of $\alpha_{M^\vee(1)}$, hence so does its dual. Again using the proposition, we obtain a second $\mathbb{Q}$-structure $B(M)$ on $\mathrm{Coker}\, \alpha_M$.

We write L^* for the leading coefficient.

Conjecture.

$$L^*(h^i(X), i+1-n)B(M) = L^*(h^i(X), n)D(M).$$

Sketch. By Poincaré duality and the hard Lefschetz theorems, we have

$$M^\vee = M(w).$$

There's then a direct calculation of

$$\frac{L^*_\infty(h^i(X), n)}{L^*_\infty(h^i(X), i+1-n)}$$

and a relationship between $B(M)/D(M)$ and the ϵ-factor worked out by Deligne based on conjectures...

3. Rational structure on absolute Hodge cohomology

For $w \leq -2$, Nekovar constructs an integral structure on $H^{i+1}_{\mathrm{MHS}^+_\mathbb{R}}$ by relating it to $\mathrm{Coker}\, \alpha_M$.

3.1. We let $\mathrm{MHS}^+_\mathbb{R}$ denote the category of mixed Hodge structures equipped with an involution ϕ_∞ preserving the weight filtration and such that $\phi_\infty \otimes c$ preserves the Hodge filtration. We let $\mathrm{MHC}^+(X)$ denote the mixed Hodge complex with involution $\in D^b(\mathrm{MHS}^+_\mathbb{R})$ associated to X. We set

$$H^i_{\mathrm{MHS}^+_\mathbb{R}}(X, \mathbb{R}(n)) := H^i R\,\mathrm{Hom}(\mathbb{R}(0), \mathrm{MHC}^+(X)(n)).$$

3.2. Proposition. Suppose $w \leq -2$. Then

$$H^{i+1}_{\mathrm{MHS}^+_{\mathbb{R}}}(X, \mathbb{R}(n)) = \operatorname{Coker} \alpha_M.$$

Proof. Recall that $H^q\mathrm{MHC}^+(X) = H^q_B(X, \mathbb{R})$. We have the second spectral sequence of Hypercohomology

$$E_2^{p,q} = \operatorname{Ext}^p_{\mathrm{MHS}^+_{\mathbb{R}}}\Big(\mathbb{R}(0), H^q_B\big(X, \mathbb{R}(n)\big)\Big) \Rightarrow H^{p+q} = H^{p+q}_{\mathrm{MHS}^+_{\mathbb{R}}}(X, \mathbb{R}(n)).$$

Since Ext^2's vanish in $\mathrm{MHS}^+_{\mathbb{R}}$, the spectral sequence degenerates to exact sequences

$$0 \to E_2^{1,i} \to H^{i+1} \to E_2^{0,i+1} \to 0.$$

For any real mixed Hodge structure with involution H, we have

$$\operatorname{Hom}(\mathbb{R}(0), H) = W_0 H^+ \cap F^0 H_{\mathbb{C}}.$$

We have

$$H \cap F^0 H_{\mathbb{C}} \subset F^0 H_{\mathbb{C}} \cap \overline{F^0 H_{\mathbb{C}}},$$

which is zero when H is pure of nonzero weight. On the other hand, if we set

$$H_{\mathrm{dR}} = H_{\mathbb{C}}^{\phi_\infty \otimes c = 1} \qquad \text{and} \qquad H^+ = H^{\phi_\infty = 1},$$

then a map

$$\operatorname{Ext}^1(\mathbb{R}(0), H) \to \frac{W_0 H_{\mathrm{dR}}}{W_0 H^+ + F^0(W_0 H_{\mathrm{dR}})}$$

is constructed as follows. Consider an extension

$$0 \to H \to E \to \mathbb{R}(0) \to 0.$$

Note that the extension splits precisely if

$$W_0 E^+ \cap F^0 E_{\mathbb{C}} \neq 0.$$

Choose arbitrarily $e \in W_0 E^+$ and $e' \in F^0 W_0 E_{\mathbb{C}}$ both mapping to $1 \in \mathbb{R}(0)$. Then we send E to $e' - e$. It is shown in Carlson, Extensions of mixed Hodge structures that this map is an isomorphism.

Putting these together, we have for $w \leq -2$,

$$H^{i+1}_{\mathrm{MHS}^+_{\mathbb{R}}}(X, \mathbb{R}(n)) = \operatorname{Ext}^1(\mathbb{R}(0), H^i_B(X, \mathbb{R}(n))) = \frac{M_{\mathrm{dR}} \otimes \mathbb{R}}{M_B^+ \otimes \mathbb{R} + F^0 M_{\mathrm{dR}} \otimes \mathbb{R}} = \operatorname{Coker} \alpha_M. \qquad \square$$

3.3. This gives us a rational structure $D(M)$ on $H^{i+1}_{\mathrm{MHS}^+_{\mathbb{R}}}(X, \mathbb{R}(n))$.

4. K-groups, regulators, the cases $w \leq -3$ and $w = -2$.

4.1. Let $X_{\mathbb{Z}}$ be a proper flat model of X. Let

$$K_{2n-i}(X)^{(n)}_{\mathbb{Z}} = \operatorname{Im}\Big(K'_{2n-i}(X_{\mathbb{Z}}) \otimes \mathbb{Q} \to K_{2n-i}(X)^{(n)}\Big).$$

This is known to be independent of choice of regular model. Beilinson conjectures this to be independent of choice of proper flat model in general (But Marc Levine recalls having seen a counterexample). We let r_H denote the regulator map

$$K_{2n-i}(X)^{(n)} \to H^i_{\mathrm{MHS}^+_{\mathbb{R}}}(X, \mathbb{R}(n)).$$

Example. $X = \operatorname{Spec} K$ a number field, $i = 0$, $n = 1$, so $w = -2$. Then Dirichle's formula corresponds to the case

$$K_{2n-(i+1)}(X)^{(n)}_{\mathbb{Z}} = K_1(\operatorname{Spec} K)^{(1)}_{\mathbb{Z}} = K_1(\mathcal{O}_K) = \mathcal{O}_K^*.$$

In the case $w = i - 2n \le -3$, the order of vanishing $\operatorname{ord}_n L(h^i(X))$ is zero, since we're inside the domain of convergence of the product.

Conjecture. Assume $w = i - 2n \le -3$. Then

$$r_H \otimes \mathbb{R} : K^{(n)}_{2n-(i+1)}(X)_{\mathbb{Z}} \otimes \mathbb{R} \to H^{i+1}_{\mathrm{MHS}^+_{\mathbb{R}}}(X, \mathbb{R}(n))$$

is an isomorphism, and

$$L^*(h^i(X), n)D(M) = \operatorname{Im} \det r_H.$$

4.2. In the case $w = i - 2n = -2$, the Tate conjecture predicts the order

$$\operatorname{ord}_n L(h^i(X)) = \dim_{\mathbb{Q}} N^{n-1}(X)$$

where N is the group of cycles modulo homological equivalence.

The cycle class map to de Rham cohomology

$$CH^n(X) \to H^{2n}_{\mathrm{dR}}(X)(n)$$

lands in $F^0 H^{2n}_{\mathrm{dR}}(X)(n) = F^n H^{2n}_{\mathrm{dR}}(X)$ because its image in $H^{2n}_B(X, \mathbb{C})$ lands in $H^{n,n}$. On the other hand, the cycle class map to Betti cohomology

$$CH^n(X) \to H^{2n}_B(X, \mathbb{Q}(n))$$

lands in the plus part. Indeed, the map

$$CH^n(X_{\mathbb{C}}) \to H^{2n}_B(X, \mathbb{Q}(n))$$

is ϕ_∞-equivariant, and

$$CH^n(X) \subset CH^n(X_{\mathbb{R}}) = CH^n(X_{\mathbb{C}})^{\phi_\infty}.$$

Set $M = h^{2n-2}(X)(n)$. Since the two cycle class maps commute with the de Rham isomorphism, we obtain a map

$$cl_B : CH^{n-1}(X) \to \operatorname{Ker} \alpha_{M(-1)}.$$

Composing with the map

$$M(-1)^+_B \otimes \mathbb{R} = M^-_B \otimes \mathbb{R} \xrightarrow{\cong} \frac{M_{\mathrm{dR}} \otimes \mathbb{R}}{M^+_B \otimes \mathbb{R}} \to \operatorname{Coker} \alpha_M,$$

we obtain a map

$$r_B : N^{n-1}(X) \to H^{2n-1}_{\mathrm{MHS}^+_{\mathbb{R}}}(X, \mathbb{R}(n))$$

from the group of cycles modulo homological equivalence.

Conjecture. Suppose $w = i - 2n = -2$. Then

$$(r_H \oplus r_B) \otimes \mathbb{R} : K^{(n)}_1(X)_{\mathbb{Z}} \otimes \mathbb{R} \oplus N^{n-1}(X) \otimes \mathbb{R} \to H^{2n-1}_{\mathrm{MHS}^+_{\mathbb{R}}}(X, \mathbb{R}(n))$$

is an isomorphism and

$$L^*(h^{2n-2}(X), n)D(M) = \operatorname{Im} \det(r_H \oplus r_B).$$

5. Appendix: Height pairing and the case $w = -1$

5.1. When $w = -1$, $M^\vee(1)$ also has negative weight, so by Proposition 2.6, α_M is an isomorphism, so $D(M)$ is a $\mathbb{Q}$-structure on $\mathbb{R}$. Instead of an Ext^1-group, $H^{2n}_{\mathrm{MHS}^+_{\mathbb{R}}}(X, \mathbb{R}(n))$ is an Ext^0-group. The regulator gets replaced by a height pairing

$$h : CH^n(X)_0 \otimes \mathbb{Q} \otimes CH^{\dim X+1-n}(X)_0 \otimes \mathbb{Q} \to \mathbb{R}.$$

Conjecture. Assume $w = -1$.

(1) The pairing h is nondegenerate.
(2) $\mathrm{ord}_n\, L(h^{2n-1}(X)) = \dim_{\mathbb{Q}} CH^n(X)_0 \otimes \mathbb{Q}$.
(3) $L^*(h^{2n-1}(X), n)D(M) = (\det h)\mathbb{Q}$.

6. The class number formula

6.1. If we consider $X = \mathrm{Spec}\,\mathbb{R}$ as a variety over $\mathbb{R}$, then the source and target of the regulator r_H associated to $h^0(X)(1)$ are given by

$$K_1^{(1)}(X) = \mathbb{R}^* \otimes \mathbb{Q}$$

and

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^0_B(X, \mathbb{R}(1))) = \frac{H^1_{\mathrm{dR}}(X)}{H^1_B(X, \mathbb{R}(1))^+} = \frac{\mathbb{R}}{\mathbb{R}(1)^+} = \mathbb{R}.$$

If we consider $\mathrm{Spec}\,\mathbb{C}$ as an $\mathbb{R}$-variety, then the source and target of r_H are given as follows. The K-group is given by

$$K_1^{(1)}(X) = \mathbb{C}^* \otimes \mathbb{Q}.$$

To compute the Ext-group we note that $H^0_B(X, \mathbb{R}(1))$ is the Hodge structure $\mathbb{R}(1)^2$ with involution

$$\phi_\infty(x, y) = (-y, -x)$$

and $H^0_{\mathrm{dR}}(X) = \mathbb{C}$. To see the embedding, we write this as

$$= (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^+ = (\mathbb{C}^2)^+.$$

So we have

$$\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^0_B(X, \mathbb{R}(1))) = \frac{(\mathbb{C}^2)^+}{(\mathbb{R}(1)^2)^+} = \frac{\mathbb{C}}{\mathbb{R}(1)} = \mathbb{R}.$$

Proposition. In both cases we have

$$r_H(x \otimes 1) = \log |x|.$$

6.2. We now consider a number field K, $X = \mathrm{Spec}\, K$ regarded as a $\mathbb{Q}$-variety, and $M = h^0(X)(1)$. The de Rham isomorphism gives rise to

$$\begin{array}{ccccccc} H^0_B(X, \mathbb{R}(1)) & \longrightarrow & H^0_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} & & & & \\ \| & & \| & & & & \\ \mathbb{R}(1)^{\mathrm{Hom}(K,\mathbb{C})} & \longrightarrow & \mathbb{C}^{\mathrm{Hom}(K,\mathbb{C})} & \longrightarrow & \mathbb{R}^{\mathrm{Hom}(K,\mathbb{C})} & \longrightarrow & 0 \end{array}$$

from which we obtain α_M and $\mathrm{Ext}^1_{\mathrm{MHS}^+_{\mathbb{R}}}(\mathbb{R}(0), H^1_B(X, \mathbb{R}(1)))$ by taking plus parts everywhere. By the proposition, the regulator

$$r_H : \mathcal{O}_K^* \otimes \mathbb{Q} \to \left(\mathbb{R}^{\mathrm{Hom}(K,\mathbb{C})}\right)^+$$

(which we've restricted to the integral K-group $K_1^{(1)}(X)_{\mathbb{Z}} = \mathcal{O}_K^* \otimes \mathbb{Q}$) is given by

$$(r_H(x \otimes 1))_\tau = \log |\tau(x)|.$$

6.3. To compute r_B we trace the generating cycle Z through the sequence of isomorphisms

$$CH^0(X) \to H_B^0(X,\mathbb{R}(0))^+ = H_B^0(X,\mathbb{R}(1))^- \otimes \mathbb{R}(-1) = \frac{H^0_{\mathrm{dR}}(X)\otimes\mathbb{R}}{H_B^0(X,\mathbb{R}(1))^+} = \operatorname{Coker}\alpha_M$$

to find that $(r_B(Z))_\tau = 1$ for all embeddings τ.

6.4. Let κ denote the leading coefficient of $\zeta_K(s) = L(h^0(X),s)$ at $s=1$. Then according to the conjecture,

$$\kappa D(M) = \operatorname{Im}\det(r_H \oplus r_B).$$

In order to compute both sides separately we equip the sequence defining $\operatorname{Coker}\alpha_M$ with natural rational structures compatible with α_M:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}(1)^{\mathrm{Hom}(K,\mathbb{C})+} & \longrightarrow & \mathbb{C}^{\mathrm{Hom}(K,\mathbb{C})+} & \longrightarrow & \mathbb{R}^{\mathrm{Hom}(K,\mathbb{C})+} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & (\mathbb{Q}i)^{\mathrm{Hom}(K,\mathbb{C})+} & \longrightarrow & (\mathbb{Q}\oplus\mathbb{Q}i)^{\mathrm{Hom}(K,\mathbb{C})+} & \longrightarrow & \mathbb{Q}^{\mathrm{Hom}(K,\mathbb{C})+} & \longrightarrow & 0 \end{array}$$

Proposition.
1. $\det M_B^+ = \pi^s \det(\mathbb{Q}i)^{\mathrm{Hom}(K,\mathbb{C})+}$.
2. $\det M_{\mathrm{dR}} = \sqrt{|d_K|}\det(\mathbb{Q}\oplus\mathbb{Q}i)^{\mathrm{Hom}(K,\mathbb{C})+}$.
3. $\operatorname{Im}\det(r_H\oplus r_B) = R\det\mathbb{Q}^{\mathrm{Hom}(K,\mathbb{C})+}$.

The Dirichlet regulator R is essentially defined by

$$\operatorname{Im}\det\lambda = R\det H_{\mathbb{Q}}$$

where $H\subset\mathbb{R}^{\mathrm{Hom}(K,\mathbb{C})+}$ is the trace zero subspace ($\mathrm{tr}(x) = \sum_\tau x_\tau$), and λ is r_H regarded as a map to H.

Corollary.

$$\kappa \equiv \frac{\pi^s}{\sqrt{|d_K|}}R \mod \mathbb{Q}^*.$$

6.5. Part (2) is the only one that isn't completely straightforward. The de Rham structure is given by

$$M_{\mathrm{dR}} = K.$$

Let $x = (x_1,\dots,x_{r+2s})$ be a basis of K over $\mathbb{Q}$. Let

$$e_\tau \in \mathbb{C}^{\mathrm{Hom}(K,\mathbb{C})}$$

denote the standard basis element associated to τ. Then a basis of $\mathbb{C}^{\mathrm{Hom}(K,\mathbb{C})}$ is given by e_ρ for ρ real, and $e_\sigma + e_{\overline{\sigma}}$, $ie_\sigma - ie_{\overline{\sigma}}$ for σ running over a set of representatives of the complex embeddings. We then have the associated matrix

$$x_i = \sum_\rho \rho(x_i)e_\rho + \sum_\sigma \operatorname{Re}\sigma(x_i)(e_\sigma + e_{\overline{\sigma}}) + \operatorname{Im}\sigma(x_i)(ie_\sigma - ie_{\overline{\sigma}}).$$

Let δ' denote its determinant. Let δ denote the determinant of the matrix

$$(\tau x_i)_{\tau,i}.$$

Then

$$d_K \equiv \delta^2 \mod \mathbb{Q}^{*2}.$$

On the other hand, a bit of linear algebra shows that

$$\frac{\delta}{\delta'} \in i^s\mathbb{Q}^*.$$

So

$$\delta' \equiv \sqrt{|\delta^2|} \equiv \sqrt{|d_K|} \mod \mathbb{Q}^*.$$

References

[Fla] Matthias Flach. The equivariant Tamagawa number conjecture: a survey. In *Stark's conjectures: recent work and new directions*, volume 358 of *Contemp. Math.*, pages 79–125. Amer. Math. Soc., Providence, RI, 2004. With an appendix by C. Greither.

[Nek] Jan Nekovář. Beĭlinson's conjectures. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 537–570. Amer. Math. Soc., Providence, RI, 1994.

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