Deligne's weight-monodromy theorem *in* Essen

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1. Lisse sheaves, constructible sheaves, weights

1.1. Let X be connected, $\bar{x}$ a geometric point. Denote by $\hat{\pi}_1$ the arithmetic fundamental group at $\bar{x}$. A representation

$$\rho : \hat{\pi}_1 \to \mathrm{GL}\, V$$

on a vector space over $\overline{\mathbb{Q}_l}$ is "l-adic" is there exists a finite extension $E \supset \mathbb{Q}_l$, an E-form V_E of E, and a factorization of ρ through an E-representation

$$\rho_E : \hat{\pi}_1 \to \mathrm{GL}(V_E)$$

which is continuous. There is then an equivalence of categories

$$\{ \text{ lisse } \overline{\mathbb{Q}_l}\text{-sheaves } \} \xrightarrow{\cong} \{l\text{-adic representations of } \hat{\pi}_1\}$$

which sends $\mathcal{F}$ to its stalk $\mathcal{F}_{\bar{x}}$ at $\bar{x}$. $\mathcal{F}$ *constructible* means that there exists a finite stratification $X = \cup X_i$ with each $\mathcal{F}|_{X_i}$ lisse.

1.2. Let $q = p^n$. Let X_0 be of finite type over $\mathbb{F}_q$, x a closed point, and $\bar{x}$ is a geometric point over x. We let F_x denote the inverse of Frobenius in $\mathrm{Gal}(k(\bar{x})/k(x))$. An element $\alpha \in \overline{\mathbb{Q}_l}$ is *pure of weight n rel. q* if it is algebraic and for all complex embeddings ι we have

$$|\iota\alpha| = q^{n/2}\,.$$

We denote by w the function $w_q(\alpha) = n$. A constructible $\overline{\mathbb{Q}_l}$-sheaf $\mathcal{F}_0$ is *(punctually) pure of weight n* if for all closed points x of X_0, the eigenvalues of F_x are pure of weight n rel. $\mathbf{N}(x) := \#(k(x))$. For the proofs, we often fix ι and talk about ι-weights, ι-pure, etc.

2. Statement of theorem

2.1. Proposition. Let V be a vector space and N a nilpotent endomorphism. Then there exists a unique increasing filtration M (we abbreviate $V_i = \mathrm{Fil}^M_i V$) such that $NV_i \subset V_{i-2}$, and such that each power N^k induces an isomorphism

$$\mathrm{gr}^M_k V \xrightarrow{\cong} \mathrm{gr}^M_{-k} V\,.$$

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Proof. This is just a verification. Here's a simple formula for M:

$$V_i = \sum_{k-j=i} \ker N^{k+1} \cap \operatorname{im} N^j \,. \qquad \square$$

2.2. Let R be a Henselian dvr with fraction field K, residue field $k \cong \mathbb{F}_q$, and π a uniformizer. We have the short exact sequence

$$0 \to I \to \operatorname{Gal}(\overline{K}/K) \to \operatorname{Gal}(\overline{k}/k) \to 0\,.$$

The action of $\operatorname{Gal}(\overline{K}/K)$ on $\mu_{l^n}(\overline{K})$ factors through $\operatorname{Gal}(\overline{k}/k)$. So the Kummer cocycle

$$\sigma \mapsto \frac{\sigma(\pi^{1/l^n})}{\pi^{1/l^n}}$$

restricts to a homomorphism

$$t_{l,n} : I \to \mu_{l^n}\,.$$

Taking inverse limits, we obtain a homomorphism

$$t_l : I \to \mathbb{Z}_l(1).$$

Now fix a generator of $\mathbb{Z}_l(1)$.

Theorem. Let V be a finite dimensional l-adic representation of $G_K = \operatorname{Gal}(\overline{K}/K)$. Then there exists an open subgroup $I_1 \subset I$ and a nilpotent endomorphism N of V such that

$$\rho(\sigma) = \exp(t_l(\sigma)N)$$

for all $\sigma \in I_1$. Moreover, the action of N commutes with the image $\rho(G_K)$ up to scalars.

This is proved by Grothendieck in SGA 7, Exp. I. If we wish to avoid fixing a generator of $\mathbb{Z}_l(1)$, we could say instead that $\rho|_{I_1}$ factors through the exponential of a nilpotent representation of the one-dimensional Lie algebra $\overline{\mathbb{Q}_l}(1)$.

2.3. In the setting of segment 2.2, the filtration associated to V and N by Proposition 2.1 is what we call the *monodromy filtration*. A final lemma before the main theorem, ensures that the *weights* of the *weight-monodromy theorem* are well-defined.

Lemma. Let V be a finite-dimensional l-adic representation of G_K, and let F', F'' be two liftings of an element F of G_k. Then the eigenvalues of F', F'' differ only by multiplication by a root of unity.

Proof. Replacing V by its semisimplification has no effect on the eigenvalues, so we may assume V is semisimple. For a reason that I'm slightly confused about at the moment, the entire representation is parabolic with respect to the filtration; semisimplicity then gives us a grading which splits the filtration. Subsequently, for each $\sigma \in I_1$, $\exp(t_l(\sigma)N)$ acts trivially. This means that $\rho|_I$ factors through a finite quotient, hence that there exists an n

such that $\rho(F')^n = \rho(F'')^n$. (An arbitrary I_1 might not be normal. I think the maximal one is; anyway, this isn't really an issue since (in any profinite group) normal finite-index subgroups form a system of neighborhoods of the identity, so I_1 contains an open normal subgroup regardless.) □

Accordingly, we may apply the terminology of segment 1.1 pertaining to absolute values of inverse Frobenius to V, by taking F to be the inverse of the Frobenius element of G_k.

2.4. Theorem. Let X_0 be a smooth curve over $\mathbb{F}_q$, $j : U_0 \hookrightarrow X_0$ open with complement S_0, s a closed point of S_0. Let $X_{0(s)}$ denote the Henselian local scheme at s, and let $\eta = \operatorname{Spec} K \in X_{0(s)}$ be its generic point. Suppose $\mathcal{F}_0$ is pure of weight β. Then for each i, the representation $\mathrm{gr}_i^M(\mathcal{F}_{0\eta})$ of G_K is pure of weight $\beta + i$.

Actually, we'll prove the stronger statement, that the same holds with "ι-pure" in place of "pure".

3. Poincaré duality / weight convergence criterion / rationality of L-functions

We need three preliminary propositions: 3.1 is a calculation based on Poincaré duality, 3.2 is a criterion for convergence of L-functions in terms of weights, and 3.3 combines the previous two with the rationality of L-functions, that is, the first part of the Weil conjectures for Lisse coefficients.

3.1. Proposition. Suppose X_0 is a curve over $\mathbb{F}_q$, and $\mathcal{F}_0$ is lisse and ι-pure of weight β. We denote base-change to $\overline{\mathbb{F}}_q$ by removing the subscript 0, and we let F denote the inverse Frobenius element of $G_{\mathbb{F}_q} = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Let α be an eigenvalue of F acting on $\mathrm{H}_c^2(X, \mathcal{F})$. Then α is ι-pure of weight $\beta + 2$.

Proof. By Poincaré duality, we have $\mathrm{H}_c^2(X, \mathcal{F}) = \mathrm{H}^0(X, \mathcal{F}^\vee(1))^\vee$. So it's enough to show that any eigenvalue of F acting on $\mathrm{H}^0(X, \mathcal{F}^\vee(1))$ is ι-pure of weight $-\beta - 2$. Since $\mathcal{F}^\vee(1)$ is lisse, the global sections map injectively into any stalk, that is, the natural map

$$\mathrm{H}^0(X, \mathcal{F}^\vee(1)) \to \mathcal{F}^\vee(1)_{\bar{x}}$$

is injective for any geometric point $\bar{x}$. Since $\mathcal{F}^\vee(1)$ is ι-pure of weight $-\beta - 2$, the proposition follows. □

3.2. Proposition. Let X_0 be of finite type over $\mathbb{F}_q$, $\mathcal{F}_0$ a constructible $\overline{\mathbb{Q}_l}$-sheaf, and let $d = \dim X$. Let $|X_0|$ denote the set of closed points, and for $x \in |X_0|$, let $\deg x$ denote the residue-field extension degree $[k(x) : \mathbb{F}_q]$. Suppose the ι-weights of $\mathcal{F}_0$ are bounded by β, that is, for every $x \in |X_0|$, every eigenvalue of F_x is ι-pure of weight $\leq \beta$. Then the product

$$\prod_{x \in |X_0|} \frac{1}{\iota \det(1 - F_x t^{\deg x}, \mathcal{F})}$$

converges, and has no zeros, in the disk $|t| < q^{-\beta/2-d}$.

By Noether normalization,

$$\#\text{ points of degree } n \leq \text{a constant} \cdot q^{dn}\,.$$

What remains is just a calculation: the product can be compared with the sum

$$\sum_n q^{nd} q^{n\beta/2} |t|^n\,.$$

3.3. Proposition. Let X_0 be a curve over $\mathbb{F}_q$, $U_0 \subset X_0$ an open subset with complement S_0. Suppose $\mathcal{F}_0$ is lisse and ι-pure of weight β on U_0. Let $s \in S_0$. Then every eigenvalue α of F_s on $j_*\mathcal{F}_0$ has $w_{\mathbf{N}(s)}(\alpha) \leq \beta$.

We may assume X_0 is affine, in which case we have

$$\prod_{x \in |U_0|} \iota \det(1 - F_x t^{\deg x}, \mathcal{F}_0)^{-1} \cdot \prod_{x \in |S_0|} \iota \det(1 - F_x t^{\deg x}, \mathcal{F}_0)^{-1} = \frac{\iota \det(1 - Ft, \mathrm{H}^1_c(X, f_*\mathcal{F}))}{\iota \det(1 - Ft, \mathrm{H}^2_c(X, f_*\mathcal{F}))}\,;$$

The first term on the left converges and doesn't vanish in the disk $|t| < q^{-(\beta+2)/2}$, and the denominator on the right doesn't vanish in the same disk. It follows that any eigenvalue α of F_s on $j_*\mathcal{F}_0$ satisfies

$$w_{\mathbf{N}(s)}(\alpha) \leq \beta + 2\,.$$

A game with tensor products enables us to replace the 2 with $2/k$ for arbitrary k.

4. Proof of theorem

After twisting by $\overline{\mathbb{Q}}_l^{(p^{-\beta/2})}$, we may assume $\beta = 0$. Here $\overline{\mathbb{Q}}_l^{(b)}$ denotes a rank-one $\overline{\mathbb{Q}_l}$-sheaf on $\operatorname{Spec} \mathbb{F}_p$ on which F (inverse Frobenius) acts by multiplication by b. (If β happens to be even, then $\overline{\mathbb{Q}}_l^{(p^{-\beta/2})} = \overline{\mathbb{Q}_l}(\beta/2)$ is a Tate object.) See paragraph 1.2.7 of Weill II.

Fix a geometric point $\overline{\eta} = \operatorname{Spec} \overline{K}$ lying over the generic point η, let $V = \mathcal{F}_{\overline{\eta}}$, and let $\rho : G_K \to \mathrm{GL}(V)$ denote the associated representation. We then have an open subgroup $I_1 \subset I$ acting through a nilpotent operator N and inducing a filtration M, the *monodromy filtration*. After possibly replacing X_0 by a finite étale cover, we may assume $I_1 = I$. We then have

$$\ker N = (\mathcal{F}_{\overline{\eta}})^I = (j_*\mathcal{F}_0)_{\overline{s}}\,.$$

By Proposition 3.3, any eigenvalue α of F_s on $\ker N$ has

$$|\iota\alpha| \leq 1\,.$$

Let $P_i := \ker(N : \mathrm{gr}_i^M V \to \mathrm{gr}_{i-2}^M V)$. The main point about the P_i's is that on the one hand, each $\mathrm{gr}_i^M V$ decomposes as a direct sum of G_K representations:

$$\text{(B)} \qquad \mathrm{gr}_i^M V = \bigoplus_{j \geq |i|, j \equiv i(2)} P_j(\frac{i+j}{2}),$$

and on the other hand, the P_i's themselves are graded pieces of $\ker N$ for the induced filtration:

$$\text{(A)} \qquad \mathrm{gr}_i^M(\ker N) = P_i .$$

Moreover, the P_i's play nicely with tensor products:

(C) $\quad P_{-j}(V) \otimes P_{-j}(V)(-j)$ is a direct summand of $P_0(V \otimes V)$,

and with duals:

$$\text{(D)} \qquad P_{-j}(V^\vee) = P_{-j}(V)^\vee(j) .$$

These facts are elementary but tedious to check.

Let α be an eigenvalue of F_s on P_{-i}. By (A) we have

$$\text{(1)} \qquad |\iota\alpha| \leq 1 .$$

By (C), $\alpha^2 q^j$ is an eigenvalue of F_s on $V \otimes V$. Apply (1) to $\mathcal{F}_0 \otimes \mathcal{F}_0$ in place of $\mathcal{F}_0$ to get $|\iota\alpha^2 q^j| \leq 1$, or, equivalently,

$$\text{(2)} \qquad |\iota\alpha| \leq q^{-j/2} .$$

By (D), $\alpha^{-1}q^{-j}$ is an eigenvalue of F_s on $P_{-j}(V^\vee)$. Apply (2) to $\mathcal{F}_0^\vee$ in place of $\mathcal{F}_0$ to get $|\iota\alpha^{-1}q^{-j}| \leq q^{-j/2}$, or equivalently,

$$|\iota\alpha| \geq q^{-j/2} .$$

So

$$|\iota\alpha| = q^{-j/2} ,$$

and we're done.

5. Scholze's statement of the theorem

5.1. Corollary. Let X_0 be a smooth open curve over $\mathbb{F}_q$, $s \in X_0(\mathbb{F}_q)$ a rational point. Let K denote the local field of X_0 at s. Let Y_0 be smooth and proper over $X_0 \setminus \{s\}$. Let $V = \mathrm{H}^i(Y_{\overline{K}}, \overline{\mathbb{Q}_l})$. Then $\mathrm{gr}_j^M V$ is pure of weight $i + j$.

Proof. Let $U_0 := X_0 \setminus \{s\}$, let f denote the map $Y_0 \to U_0$, and let j denote the open immersion $U_0 \hookrightarrow X_0$. Let $\mathcal{F}_0 = R^i f_* \overline{\mathbb{Q}_l}$. Then $\mathcal{F}_0$ is lisse and pure of weight i. Moreover, $\mathrm{H}^i(Y_{\overline{K}}, \overline{\mathbb{Q}_l})$ is equal to the stalk $(j_*\mathcal{F}_0)_{\overline{\eta}}$ at a geometric local point $\overline{\eta} = \mathrm{Spec}\,\overline{K}$ over $\eta = \mathrm{Spec}\,K$ by proper base-change. So this follows from the theorem. $\square$

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