K-GROUPS AND SECONDARY INVARIANTS

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1. Chern classes and étale cycle class maps for K_0

These notes are mostly about higher K-groups and their Chern class maps all the way to étale cohomology, skipping the higher Chow groups which lie in between. However, I'll start by mentioning ordinary Chow groups briefly, since these help to clarify the relationship between K-theory and *absolute cohomology*.

1.1. Let $Z = \operatorname{Spec} k$ a field and X smooth over Z. Let $\Lambda = \mathbb{Z}/l^n$ for some n which we fix and promptly forget (so in particular, this n need not be equal to any other n that occurs bellow). The Kummer sequence of sheaves on $\mathrm{Sm}_{Z,\text{ét}}$

$$1 \to \Lambda(1) \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

gives rise to a map

$$\operatorname{Pic}(X) = H^1_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{\sigma} H^2_{\text{ét}}(X, \Lambda(1)).$$

Choosing an algebraic closure, there's a further map

$$H^1_{\text{ét}}(X, \Lambda(1)) \xrightarrow{f} H^2_{\text{ét}}(X_{\bar{k}}, \Lambda(1))^{G_k}.$$

The image of the composite $f \circ \sigma$ is by definition the group of divisors modulo *homological equivalence*, which we will denote by $\hom^1(X)$. (When X is proper, $\hom^1(X)$ is also called the Néron-Severi group, and is closely related to the maximal étale quotient of the Picard variety.)

Although we all learn about the Picard group before learning about $\hom^1$, the general picture suggests that psychologically, it makes some sense to reverse the order. The *cycle class map*

$$\hom^1(X) \xrightarrow{\overline{\sigma}} H^2_{\text{ét}}(X_{\bar{k}}, \Lambda(1))^{G_k}$$

should be viewed as psychologically prior. A partial justification for this is that homological equivalence is conjecturally equivalent to *numerical equivalence*, while the latter may be defined without reference to any Weil cohomology theory.

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The map σ is then a refinement of the *cycle class map* $\overline{\sigma}$ whose target is sometimes described as *absolute* cohomology. More on that later.

1.2. The situation described in paragraph 1.1 is the case $j = 1$ of the square

$$\begin{array}{ccc} \mathrm{Ch}^j(X) & \longrightarrow & H^{2j}(X, \Lambda(j)) \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ \hom^j(X) & \longrightarrow & H^{2j}(X_{\bar{k}}, \Lambda(j))^{G_k}. \end{array}$$

The Λ-module occurring in the south-eastern corner of the diagram is a cohomology group (H^0) of a Λ-adic Galois representation. The category of such, $\mathbf{Rep}_\Lambda\, G_k$, is abelian, so we can form its derived category $D\,\mathbf{Rep}_\Lambda\, G_k$, which comes together with a t-structure whose heart is

$$\mathbf{Rep}_\Lambda\, G_k \subset D\,\mathbf{Rep}_\Lambda\, G_k.$$

Over the years (especially in the context of motives) it has become clear that it's better to think of the derived category $D\,\mathbf{Rep}_\Lambda\, G_k$ as being psychologically prior, and of the abelian category $\mathbf{Rep}_\Lambda\, G_k$ as resulting from extra structure, namely a t-structure. So it's worth noting that the *absolute cohomology* occurring in the north-eastern corner of the diagram can be defined without reference to the t-structure. To see this, denote the structure map $X \to Z$ by g and consider the derived pushforward

$$C = C^*_{\text{ét}}(X, \Lambda(j)) := Rg_*\Lambda(j).$$

Then we have canonical isomorphisms

$$\begin{aligned} H^k(X, \Lambda(j)) &= h^k R\Gamma C \\ &= h^k \underline{RHom}(\Lambda(0), C) \\ &= R\,\mathrm{Hom}(\Lambda(0), C[2j]), \end{aligned}$$

where $\underline{RHom}$ denotes the *mapping complex*, an object of the derived category of Λ-modules. Thus, the Galois fixed part of the *geometric cohomology* $H^{2j}(X_{\bar{k}}, \Lambda(j)) = H^{2j}(X_{\bar{k}}, \Lambda)(j)$ is a *piece* of the absolute cohomology which may be broken-off with the help of the t-structure.

1.3. Thus, rational equivalence classes of cycles give rise to absolute cohomology classes. On the other hand, vector bundles give rise to cycles which are well defined modulo rational equivalence. Starting now, however, and for the remainder of these notes, we will skip Chow groups and go straight from vector bundles (and the K-groups to which they give rise) to absolute cohomology.

Let E be a vector bundle of rank $r + 1$ on X and let $\mathbb{P}E$ denote its projectivization. The Kummer sequence as above applied to $\mathbb{P}E$ in place of X

yields a map

$$\operatorname{Pic}\mathbb{P}E = H^1_{\text{ét}}(\mathbb{P}E, \mathbb{G}_m) \xrightarrow{\tilde{c}} H^2_{\text{ét}}(\mathbb{P}E, \Lambda(1));$$

we set

$$\xi_E := \tilde{c}(\mathcal{O}(1)).$$

1.4. If Y is a smooth Z-scheme, we define

$$H^*_{\text{ét}}(Y) := \bigoplus_j H^{2j}(Y_{\bar{k}}, \Lambda(j)).$$

Then cup product makes $H^*_{\text{ét}}(Y)$ into a commutative Λ-algebra (since only even degrees appear) and the map

$$\pi^* : H^*_{\text{ét}}(X) \to H^*_{\text{ét}}(\mathbb{P}E)$$

makes $H^*_{\text{ét}}(\mathbb{P}E)$ into a $H^*_{\text{ét}}(X)$-algebra.

Theorem 1.4.1. The algebra $H^*_{\text{ét}}(\mathbb{P}E)$ is freely generated as a $H^*_{\text{ét}}(X)$-module by the elements $1, \xi_E, \xi_E^2, \ldots, \xi_E^r$.

In different notation, identifying $H^{2j}(X, \Lambda(j))$ with its image in $H^{2j}(\mathbb{P}E, \Lambda(j))$, we have an isomorphism of $H^*(X)$-modules

$$H^*(\mathbb{P}E) \simeq H^*(X) \oplus H^*(X)\xi_E \oplus \cdots \oplus H^*(X)\xi_E^r.$$

Taking jth graded components (and dropping the free generators from the notation), we get

$$H^{2j}(\mathbb{P}E, \Lambda(j)) \simeq H^{2j}(X, \Lambda(j)) \oplus H^{2j-2}(X, \Lambda(j-1)) \oplus \cdots \oplus H^{2j-2r}(X, \Lambda(j-r)),$$

where any terms outside of the range of dimensions of X vanish.

In view of Theorem 1.4.1, we may define the Chern classes

$$c_j(E) \in H^{2j}(Y, \Lambda(j))$$

of a vector bundle E by the formula

$$\xi_E^{r+1} + \pi^*(c_1(E)) \cup \xi_E^r + \cdots + \pi^*(c_{r+1}(E)) = 0 \quad \text{in} \quad H^{2r+2}(\mathbb{P}E, \Lambda(r+1)).$$

Put more simply (while being less explicit about the Tate twists which show up), we write ξ_E^{r+1} as a linear combination of the generators $1, \xi_E, \ldots, \xi_E^r$ and define the Chern classes to be the coefficients (or minus the coefficients, or an alternating sum — this seems to vary depending on the source).

1.5. Denote by $\mathrm{Proj}_X^{\simeq}$ the groupoid of $\mathcal{O}_X$-modules which are locally free of finite rank and isomorphisms of $\mathcal{O}_X$-modules. Direct sum makes $\mathrm{Proj}_X^{\simeq}$ into a symmetric monoidal category. This makes $\pi_0\mathrm{Proj}_X^{\simeq}$ into a commutative monoid. We let

$$\pi_0\mathrm{Proj}_X^{\simeq} \to K_0(X)$$

be the universal map to an abelian group. For each $j \geq 0$, there's a map

$$K(X)_0 \to \mathrm{Ch}^j(X),$$

which, I believe, may be defined by

$$c_j([E]-[F]) = c_j(E) - c_j(F).$$

That is, each c_j is a map of monoids, so extends uniquely to a map of groups.

2. From group completion to the plus construction

2.1. If X is a simplicial scheme over a base Z, then a vector bundle on X consists of for every $I \in \Delta$, a vector bundle E_I on X_I, and for every map $I \to J$ in Δ inducing $f : X_I \leftarrow X_J$, an isomorphism

$$f^* E_I \xrightarrow{\sim} E_J.$$

These form a category in a natural way.

2.2. We collect here some facts about BG; some will be used in this section, others in the next section.

If G is an abstract group, BG can be thought of first as the category with one objects whose automorphisms are given by G. It's often better to think of BG as a simplicial set by taking the nerve of this category, or as a topological space, by taking geometric realization of the simplicial set. We won't distinguish between these notationally.

More generally, one can do "the same" with a sheaf of groups G (on a topological space or a site) to obtain a simplicial sheaf BG. Here, along the way, one may naively be led to considering a sheaf of categories up to equality of functors, rather than up to equivalences together with coherent higher homotopies. I'm not sure if this is dangerous.

In any case, if G is an algebraic group over a base Z, we will think of BG as a simplicial Z-scheme. There's a map $Z \to BG$ and the trivial action of G on Z over BG makes $Z \to BG$ into a very special G-torsor[1]. Since GL_n-torsors correspond to vector bundles, BGL_n has a very special vector bundle E_n^u of rank n.[2]

We set

$$GL = \operatorname{colim} GL_n.$$

2.3. Let X be a scheme. Since $\mathrm{Proj}_X^{\simeq}$ is a groupoid, its nerve $N(\mathrm{Proj}_X^{\simeq})$ is a Kan-complex, i.e. an object of the ∞-category $\mathcal{S}$ of *spaces*. The symmetric

[1] that is, it's the *universal* torsor in a sense which can be made precise in several different ways.

[2] I believe the universality may be (partly) described in down to earth terms as follows. Given a scheme Y and a vector bundle of rank n on Y, there exists a Zariski hypercover $h : Y_\bullet \to Y$ and a map $f_E : Y_\bullet \to BGL_n$ such that $f_E^* E_n^u$ includes descent data for E along h.

monoidal structure gives $N(\mathrm{Proj}_X^{\simeq})$ the structure of an E_∞-monoid in $\mathcal{S}$. We let

$$N(\mathrm{Proj}_X^{\simeq}) \to K(X)$$

be the universal map to an E_∞-group. [3] There's an equivalence of ∞-categories

$$\mathrm{Sp}_{\geq 0} \xrightarrow{\simeq} \mathrm{Grp}_{E_\infty}(\mathcal{S})$$

from connective spectra to E_∞-groups, so that $K(X)$ may be canonically identified with a spectrum. [4] It follows formally from an ∞-categorical adjunction that

$$\pi_0(K(X)) \simeq K_0(X).$$

2.4. Assume now that $X = \operatorname{Spec} R$ is affine. For each $n \geq 0$, there's a natural fully faithful functor

$$BGL_n(R) \to \mathrm{Proj}_R^{\simeq}$$

with essential image equivalent to the full subcategory of free modules of rank n. Hence there's a map of simplicial sets

$$BGL_n(R) \to N(\mathrm{Proj}_R^{\simeq}) \to K(R).$$

For varying n, the map

$$BGL_n(R) \to BGL_{n+1}(R)$$
$$g \mapsto g \oplus (1)$$

(direct sum of matrices) commutes with the map

$$\mathrm{Proj}_R^{\simeq} \to \mathrm{Proj}_R^{\simeq}$$
$$E \mapsto E \oplus R.$$

The latter becomes invertible upon passage to $K(R)$. Taking direct limits, we obtain (or at least expect) a map

$$BGL(R) \to K(R).$$

There's also a natural map

$$K(R) \to \pi_0 K(R) = K_0(R).$$

This last map admits a section, so that there exists a map

$$f : K_0(R) \times BGL(R) \to K(R).$$

[3] An E_∞-*monoid* is a space with a distinguished point and a binary operation which are unital, associative and commutative up to coherent higher homotopy. An E_∞-*group* is an E_∞-monoid such that the induced monoid structure on π_0 has inverses.

[4] In more down to earth terms, this means that the underlying space of $K(X)$ comes with an infinite sequence of deloopings; these are spaces $B^n K(X)$ together with homotopy equivalences $\Omega^n B^n K(X) \simeq K(X)$.

Theorem 2.4.1 (The plus theorem)**.** Perhaps under some assumptions on R, f induces abelianization on π_1 and isomorphisms on homology with integral (as well as some other) coefficients.

Lectures by Thomas Nikolaus at the Isaac Newton Institute available from `https://www.newton.ac.uk/event/hhhw01/` are crystal clear, until they become suddenly a bit vague at exactly this point, somewhere in the middle of lecture II. Most of the work is done in his *The group completion theorem via localizations of ring spectra*; see also the exposition *Group completion is a completion* by Oscar Bendix Harr available from `http://web.math.ku.dk/~jg/students/oscar-bsthesis.pdf`, as well as notes by Shay Ben Moshe available from
`http://shaybm.com/k-theory-seminar/`.

The proof of Theorem 2.4.1 goes through a certain modification $BGL(R)^+$ of the space $BGL(R)$, commonly referred to as the *plus construction*. My formulation of the theorem avoids mentioning it. See the notes by Ben Moshe for an informal account of how the plus construction solves the group completion problem in two steps.

2.5. As a corollary, $K_1(X) = R^*$. Indeed,

$$\begin{aligned} K_1(X) &= \pi_1 K(X) \\ &= (\pi_1 BGL(R))^{\mathrm{ab}} \\ &= GL(R)^{\mathrm{ab}}, \end{aligned}$$

and a calculation shows that the latter maps isomorphically to R^* via the determinant.

3. Chern class maps

Let X be a smooth quasiprojective k-scheme and $\Lambda = \mathbb{Z}/l^n$ (l invertible in k). Our goal in this section is to construct the Chern class maps

$$c_{i,j} : K_i(X) \to H^{2j-i}_{\text{ét}}(X, \Lambda(j)).$$

These notes are based on Peter Schneider: *Introduction to the Beilinson conjectures*

3.1. Let $Z = \operatorname{Spec} k$, a field. Let $\Lambda = \mathbb{Z}/l^m$, l different from the characteristic of k. Let

$$\mathcal{A} = \bigoplus_{j \geq 0} \Lambda(j)$$

an étale sheaf on the category Sm_Z of smooth quasiprojective schemes over Z. Let a be the map

$$\mathrm{Sm}_{Z,\text{ét}} \to \mathrm{Sm}_{Z,\mathrm{Zar}}\,.$$

Let $\mathcal{B} = Ra_*\mathcal{A}$, an object of the derived category

$$h\mathcal{D}^+ = h\mathcal{D}^+(\mathrm{Sm}_{Z,\mathrm{Zar}}, \mathrm{Ab})$$

of bounded below complexes of sheaves of abelian groups. Let C be a quasi-isomorphic complex of injectives.

The object $\mathcal{A}$ of the bounded below derived category $h\mathcal{D}^+_{\text{ét}}$ has a natural group structure. With some work, this leads to associated group structures on $\mathcal{B}$ and on C. This can make things simpler, although I doubt we really need it here.

3.2. For G an abstract group, we have

$$H^{-i}N\mathbb{Z}BG = H_i(BG, \mathbb{Z}) = H_i(G, \mathbb{Z}).$$

This is just the same as the usual way of computing group homology as the homology of an explicitly constructed complex.

3.3. Let Y be a simplicial object of Sm_Z. We denote by $\mathbb{Z}Y$ the simplicial sheaf of free abelian groups generated by the functor of points of Y. We let N of a simplicial abelian group denote the associated complex. Let $\mathcal{F}$ be an object of the derived category

$$h\mathcal{D}_{\mathrm{Zar}}(\mathrm{Sm}_Z, \mathrm{Ab}).$$

We define

$$H^i(Y, \mathcal{F}) = \mathrm{Hom}_{h\mathcal{D}}(N\mathbb{Z}Y, \mathcal{F}[i]).$$

Of course, the notation implies that where the left-hand side is already defined in a different way, the two agree.

3.4. Roughly speaking, c_j of a vector bundle is the degeneracy locus of a generic frame of appropriate size. The construction can be made sufficiently functorial to work also for simplicial schemes. We give a very superficial account.

The Kummer extension

$$0 \to \Lambda(1) \to \mathbb{G}_m \to \mathbb{G}_m \to 1$$

in the category of étale sheaves of abelian groups on Sm_Z gives rise to a map

$$\mathbb{G}_m[-1] \to \Lambda(1) \to \mathcal{A}$$

in the derived category of étale sheaves of abelian groups, hence to a map of complexes of Zariski sheaves [5]

$$\tilde{c} : \mathbb{G}_m[-1] \to C.$$

[5] Actually, Schneider merely hints at the existence of a map $\tilde{c}$ for which Theorem 3.4.1 holds. My construction may be wrong.

Let Y be a simplicial object of Sm_Z and E a vector bundle. Then $\tilde{c}$ induces a map
$$\tilde{c}_E : H^1(\mathbb{P}E, \mathbb{G}_m) \to H^2(\mathbb{P}E, C).$$
We denote
$$\xi_E := \tilde{c}_E(\mathcal{O}(1)).$$

Theorem 3.4.1. Let Y be a simplicial object of Sm_Z and E a vector bundle of rank $n+1$. Denote the projection $\mathbb{P}E \to Y$ by π. Then for each i, the map
$$\bigoplus_{k=0}^{n} H^{i-2k}(Y, C) \to H^i(\mathbb{P}E, C)$$
$$(\alpha_0, \dots, \alpha_n) \mapsto \pi^*\alpha_0 + \pi^*\alpha_1 \cup \xi_E + \dots + \pi^*\alpha_n \cup \xi_E^n$$
is an isomorphism.

In view of Theorem 3.4.1, we may define the Chern classes
$$c_j(E) \in H^{2j}(Y, C)$$
of a vector bundle E by the formula
$$\xi_E^{n+1} + \pi^*(c_1(E)) \cup \xi_E^n + \dots + \pi^*(c_{n+1}(E)) = 0 \quad \text{in} \quad H^{2n+2}(\mathbb{P}E, C).$$
We set $c_0(E) = 1$ and $c_{>n+1}(E) = 0$.

3.5. We define
$$c_j^{(n)} \in H^{2j}(BGL_n, \mathrm{C})$$
by [6]
$$c_j^{(n)} = c_j\big([E_n^u] - [\mathcal{O}^n]\big).$$
These arrange themselves into classes
$$c_j \in H^2(BGL, C)$$
represented by maps
$$N\mathbb{Z}BGL \to C[2j].$$

3.6. Let $X = \operatorname{Spec} A$ be an affine scheme in Sm_Z. Evaluating c_j on X we obtain
$$c_j(X) : N\mathbb{Z}BGL(X) \to C[2j](X)$$
hence
$$H^{-i}(c_j(X)) : H_i(GL(A), \mathbb{Z}) \to H^{2j-i}(X, C) = H^{2j-i}_{\text{ét}}(X, \bigoplus_{j \geq 0} \Lambda(j)).$$

[6] I haven't quite understood this formula yet; I find Schneider's account confusing here.

We define $c_{i,j}$ for $i \geq 1$ by the composite

$$\begin{tikzcd}
K_i(X) = \pi_i(K(A)) \arrow[r] & H_i(K(A), \mathbb{Z}) \arrow[d, equal] & \\
& H_i(BGL(A), \mathbb{Z}) \arrow[d, equal] & \\
& H_i(GL(A), \mathbb{Z}) \arrow[r] & H^{2j-i}_{\text{ét}}(X, \bigoplus_{k\geq 0} \Lambda(k)) \arrow[d] \\
& & H^{2j-i}_{\text{ét}}(X, \Lambda(j)).
\end{tikzcd}$$

Theorem 3.7. If $W \to Y$ is a morphism in Sm_Z which is Zariski locally projection from affine space then the induced map

$$K(W) \leftarrow K(Y)$$

is a homotopy equivalence.

3.8. By Jouanolou's lemma, every $Y \in \mathrm{Sm}_Z$ has an affine W mapping to Y via a map as in the theorem. Since such maps also induce isomorphisms in étale cohomology, this may be used to extend the Chern class maps to all of Sm_Z. Schneider comments on a more sophisticated and conceptually correct method which applies also to cohomology theories which are not homotopy invariant.

4. Secondary invariants

4.1. The functors

$$\mathrm{Ab}(X_{\text{ét}}) \xrightarrow{f_*} \mathrm{Ab}(Z_{\text{ét}}) \xrightarrow{\Gamma} \mathrm{Ab}$$

give rise to a spectral sequence

$$R^q\Gamma R^p f_*\Lambda(j) \Longrightarrow R^{p+q}\Gamma\Lambda(j).$$

This induces a filtration on $H^{2j-i}_{\text{ét}}(X, \Lambda(j))$ and, for instance, a short exact sequence

$$0 \to H^1\Big(G_k, H^{2j-i-1}_{\text{ét}}(X_{\bar{k}}, \Lambda(j))\Big) \to H^{2j-i}_{\text{ét}}(X, \Lambda(j))/F^2 \to H^0\Big(G_k, H^{2j-i}_{\text{ét}}(X_{\bar{k}}, \Lambda(j))\Big) \to 0.$$

Thus the kernel of

$$K_i(X) \to H^0\Big(G_k, H^{2j-i}_{\text{ét}}(X_{\bar{k}}, \Lambda(j))\Big)$$

maps to

$$H^1\Big(G_k, H^{2j-i-1}_{\text{ét}}(X_{\bar{k}}, \Lambda(j))\Big),$$

providing a secondary invariant for classes which are *homologically trivial*.

Example 4.2. Suppose X/k is a smooth proper curve. Then Poincaré duality provides an isomorphism

$$H^1(X_{\bar{k}}, \Lambda)(1) \simeq H_1(X_{\bar{k}}, \Lambda),$$

the Λ-adic Tate module of the Jacobian. Hence the kernel $K_0(X)_0$ of

$$K_0(X) \to H^0\Big(G_k, H^2_{\text{ét}}(X_{\bar{k}}, \Lambda(1))\Big)$$

maps to

$$H^1\Big(G_k, H^1_{\text{ét}}(X_{\bar{k}}, \Lambda(1))\Big) = H^1\Big(G_k, H_1^{\text{ét}}(X_{\bar{k}}, \Lambda)\Big).$$