Explicit Chabauty-Kim theory for the thrice punctured line

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joint work with Stefan Wewers

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Let

\[ X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \].

For any ring (commutative with unit), the set \( X(R) \) of “\( R \)-valued points of \( X \)” is equal to the set

\[ \{(x, y) \in R^* \times R^* \mid x + y = 1\} \]

of solutions to the unit equation in \( R \).

Let

\[ S = \{q_1, \ldots, q_s\} \]

be a finite set of primes.
Siegel’s theorem tells us that $X(\mathbb{Z}[S^{-1}])$ is finite.

Baker’s method gives us bounds on the number of points, which are typically large.

But we’re still not very good at computing the sets $X(\mathbb{Z}[S^{-1}])$. 
In 2005 Minhyong Kim began to develop a new approach to the study of integral points, inspired by Chabauty’s method. After fixing a prime

$$p \notin S,$$

Kim constructs a pair of towers of $\mathbb{Q}_p$-varieties and morphisms between them like so,

\[
\begin{array}{ccc}
\vdots & \downarrow & \vdots \\
\text{Sel}_{2,\mathbb{Q}_p} & \xrightarrow{h_2} & \text{Alb}_2 \\
\downarrow & & \downarrow \\
\text{Sel}_{1,\mathbb{Q}_p} & \xrightarrow{h_1} & \text{Alb}_1 \\
\end{array}
\]
and for each $n$, a commuting square like so.

\[
\begin{array}{ccc}
X(\mathbb{Z}[S^{-1}]) & \to & X(\mathbb{Z}_p) \\
\kappa \downarrow & & \downarrow \alpha \\
\text{Sel}_n(\mathbb{Q}_p) & \to & \text{Alb}_n(\mathbb{Q}_p) \\
\downarrow h & & \\
h_n & & \\
\end{array}
\]
We obtain a nested sequence of subsets of $X(\mathbb{Z}_p)$ containing $X(\mathbb{Z}[S^{-1}])$

$$X(\mathbb{Z}_p) \supset \alpha^{-1} h_1 \text{Sel}_1(\mathbb{Q}_p) \supset \alpha^{-1} h_2 \text{Sel}_2(\mathbb{Q}_p) \supset \cdots \supset X(\mathbb{Z}[S^{-1}]).$$

Kim conjectures that for $n$ large, we have equality:

$$\alpha^{-1} h_n \text{Sel}_n(\mathbb{Q}_p) = X(\mathbb{Z}[S^{-1}]).$$

Wewers and I want to compute these subsets.

Our contribution so far concerns $h_2$. 
Let $H$ denote the group

$$
\begin{pmatrix}
1 & * & * \\
* & 1 & * \\
* & * & 1
\end{pmatrix}
$$

of upper triangular $3 \times 3$ matrices. We construct a diagram like so.

$$
\begin{align*}
X(\mathbb{Z}[S^{-1}]) \xrightarrow{\kappa} X(\mathbb{Z}_p) \\
\mathbb{Q}^S \times \mathbb{Q}^S \xrightarrow{h} H(\mathbb{Q}_p)
\end{align*}
$$

\[(*\)]
We define \( \kappa \) by

\[
\kappa(x, y) = (x_1, \ldots, x_s; y_1, \ldots, y_s)
\]

where

\[
\begin{align*}
x &= \pm q_1^{x_1} \cdots q_s^{x_s} \\
y &= \pm q_1^{y_1} \cdots q_s^{y_s}
\end{align*}
\]
We define $\alpha$ by

$$\alpha(x, y) = \begin{pmatrix} 1 & \log x & -\text{Li}_1 x \\ 0 & 1 & \log y \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Here $\text{Li}_1$ denotes the $p$-adic dilogarithm.
To define $h$, Write

$$h = \begin{pmatrix} 1 & h_{1,2} & h_{1,3} \\ 1 & h_{2,3} & 1 \end{pmatrix}.$$

Then the first two components are linear, given by

$$h_{1,2}(x, y) = (\log q_1)x_1 + \cdots + (\log q_s)x_s$$
$$h_{2,3}(x, y) = (\log q_1)y_1 + \cdots + (\log q_s)y_s.$$
Now let $E = \mathbb{Q} \otimes \mathbb{Q}^*$ regarded as a $\mathbb{Q}$ vector space, written multiplicatively. Then for each $i, j$, the generator $q_i \otimes q_j$ of $E \otimes E$ may be written in the form

$$q_i \otimes q_j = \prod_k ((u_k \otimes v_k)(v_k \otimes u_k))^{s_k} \cdot \prod_l (t_l \otimes (1 - t_l))^{d_l}.$$ 

Indeed, the quotient by elements of this form is Matsumoto’s presentation for $K_2$, and by Tate’s calculation, we have $K_2(\mathbb{Q}) \otimes \mathbb{Q} = 0$. Moreover, the proof amounts to a simple algorithm for producing such an expression. In terms of these decompositions, $h_{1,3}$ is bilinear given by

$$h_{1,3}(x, y) = \sum a_{i,j} x_i y_j$$

where

$$a_{i,j} = \sum_k s_k (\log u_k)(\log v_k) - \sum_l d_l (\text{Li} \ t_l).$$
Theorem (DC, Wewers). The diagram

\[
\begin{array}{ccc}
X(\mathbb{Z}[S^{-1}]) & \longrightarrow & X(\mathbb{Z}_p) \\
\downarrow \kappa & & \downarrow \alpha \\
\mathbb{Q}S \times \mathbb{Q}S & \underset{h}{\longrightarrow} & H(\mathbb{Q}_p)
\end{array}
\]

commutes.
Example

Let $S = \{2\}$. Then

$$h(x, y) = \begin{pmatrix} 1 & (\log 2)x & \frac{1}{2}(\log 2)^2 xy \\ 1 & 1 & (\log 2)y \\ 1 & 1 & 1 \end{pmatrix}$$

so its image satisfies the equation

$$XY = 2Z$$

which pulls back to

$$\log(x) \log(y) = -2 \text{Li}(x)$$

on $X(\mathbb{Z}_p)$. 
Now set $p = 3$. Then computations based on the work of Besser–de Jeu show that the roots are precisely $\{-1, 1/2, 2\}$.

The same holds for $p = 5, 7$.

For $p = 11$ you get more roots,

and after that the number of roots seems to increase quickly.
Proof

The determination of $h_{1,2}, h_{2,3}$, is a direct calculation in (abelian) $p$-adic Hodge theory, and was previously known. We turn our attention to $h_{1,3}$. 
The category of mixed Tate motives over $\mathbb{Q}$ with $\mathbb{Q}$-coefficients unramified outside of $S$, is canonically

$$\text{Rep } B,$$

where

$$B = \mathbb{G}_m \ltimes U,$$

where $U$ is a prounipotent $\mathbb{Q}$-group. Its coordinate ring

$$A = \text{regular functions on } U$$

has the structure of a graded Hopf algebra.
The $p$-adic de Rham realization of a mixed Tate motive has the structure of a mixed Tate filtered $\varphi$ module.

The category of mixed Tate filtered $\varphi$ modules has a similar structure;

we decorate with $\text{dR}$. 
We construct a diagram like so.

\[ X(\mathbb{Z}[S^{-1}]) \longrightarrow X(\mathbb{Z}_p) \]

\[ \bar{\kappa} \downarrow \quad \bar{\alpha} \downarrow \]

\[ A_2 \quad \rho \longrightarrow A_2^{dR} \quad \psi \longrightarrow \mathbb{Q}_p \]

\[ \nu \downarrow \cong \downarrow \]

\[ A_1 \otimes A_1 \]

\[ \mathbb{Q}^S \otimes \mathbb{Q}^S \]
The theory of the unipotent fundamental group assigns to every point \((x, y) \in X(\mathbb{Z}[S^{-1}])\) a \(B\)-equivariant \(H\)-torsor. By the general theory of mixed Tate categories, such an object gives rise to an element \(\bar{\kappa}(x) \in A_2\).

A similar description applies to \(\bar{\alpha}\).

Commutativity of the square follows from Olsson’s unipotent \(p\)-adic Hodge theory.

As for \(\Psi\), let me just say vaguely that it comes from comparing the slope decomposition with the Hodge filtration.
Tate’s decomposition into elements of the form

\[ u \otimes v + v \otimes u \]  \hspace{1cm} (D)

and elements of the form

\[ t \otimes (1 - t) \]  \hspace{1cm} (G)

takes place in the lower left (after possibly enlarging \( S \)). Elements of type \( G \) ("geometric elements") come from \( X \), and explicit formulas for \( \alpha \) obtained by Furusho deliver the answer

\[ h_{1,3}(t \otimes r) = - \text{Li}(t). \]

On the other hand, elements of type \( D \) are in a certain sense *decomposable*, so the computation of their image reduces to the abelian case.