Explicit Chabauty-Kim theory for the thrice punctured line

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joint work with Stefan Wewers

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Introduction

Let

$$X = \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

For any ring (commutative with unit), the set $X(R)$ of "R-valued points of X" is equal to the set

$$\left\{(x, y) \in R^* \times R^* \;\middle|\; x + y = 1\right\}$$

of solutions to the unit equation in R.
Let

$$S = \{q_1, \ldots, q_s\}$$

be a finite set of primes.

- Siegel's theorem tells us that $X(\mathbb{Z}[S^{-1}])$ is finite.
- Baker's method gives us bounds on the number of points, which are typically large.
- But we're still not very good at computing the sets $X(\mathbb{Z}[S^{-1}])$.

In 2005 Minhyong Kim began to develop a new approach to the study of integral points, inspired by Chabauty's method. After fixing a prime

$$p \notin S\,,$$

Kim constructs a pair of towers of $\mathbb{Q}_p$-varieties and morphisms between them like so,

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathrm{Sel}_{2,\mathbb{Q}_p} & \xrightarrow{h_2} & \mathrm{Alb}_2 \\ \downarrow & & \downarrow \\ \mathrm{Sel}_{1,\mathbb{Q}_p} & \xrightarrow{h_1} & \mathrm{Alb}_1 \end{array}$$

and for each n, a commuting square like so.

$$\begin{array}{ccc} X(\mathbb{Z}[S^{-1}]) & \longrightarrow & X(\mathbb{Z}_p) \\ \kappa \downarrow & & \downarrow \alpha \\ \mathrm{Sel}_n(\mathbb{Q}_p) & \xrightarrow[h_n]{} & \mathrm{Alb}_n(\mathbb{Q}_p) \end{array}$$

We obtain a nested sequence of subsets of $X(\mathbb{Z}_p)$ containing $X(\mathbb{Z}[S^{-1}])$

$$X(\mathbb{Z}_p) \supset \alpha^{-1} h_1 \mathrm{Sel}_1(\mathbb{Q}_p) \supset \alpha^{-1} h_2 \mathrm{Sel}_2(\mathbb{Q}_p) \supset \cdots \supset X(\mathbb{Z}[S^{-1}])\,.$$

Kim conjectures that for n large, we have equality:

$$\alpha^{-1} h_n \mathrm{Sel}_n(\mathbb{Q}_p) = X(\mathbb{Z}[S^{-1}])\,.$$

Wewers and I want to compute these subsets.

Our contribution so far concerns h_2.

Statement

Let H denote the group

$$\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

of upper triangular 3×3 matrices. We construct a diagram like so.

$$\begin{array}{ccc} X(\mathbb{Z}[S^{-1}]) & \longrightarrow & X(\mathbb{Z}_p) \\ \kappa \downarrow & & \downarrow \alpha \\ \mathbb{Q}^S \times \mathbb{Q}^S & \xrightarrow[h]{} & H(\mathbb{Q}_p) \end{array} \qquad (*)$$

We define κ by

$$\kappa(x, y) = (x_1, \ldots, x_s; y_1, \ldots, y_s)$$

where

$$\begin{aligned} x &= \pm q_1^{x_1} \cdots q_s^{x_s} \\ y &= \pm q_1^{y_1} \cdots q_s^{y_s} \, . \end{aligned}$$

We define α by

$$\alpha(x, y) = \begin{pmatrix} 1 & \log x & -\operatorname{Li} x \\ 0 & 1 & \log y \\ 0 & 0 & 1 \end{pmatrix}.$$

Here Li denotes the p-adic dilogarithm.

To define h, Write

$$h = \begin{pmatrix} 1 & h_{1,2} & h_{1,3} \\ & 1 & h_{2,3} \\ & & 1 \end{pmatrix} .$$

Then the first two components are linear, given by

$$\begin{aligned} h_{1,2}(x, y) &= (\log q_1)x_1 + \cdots + (\log q_s)x_s \\ h_{2,3}(x, y) &= (\log q_1)y_1 + \cdots + (\log q_s)y_s . \end{aligned}$$

Now let $E = \mathbb{Q} \otimes \mathbb{Q}^*$ regarded as a $\mathbb{Q}$ vector space, written multiplicatively. Then for each i, j, the generator $q_i \otimes q_j$ of $E \otimes E$ may be written in the form

$$q_i \otimes q_j = \prod_k ((u_k \otimes v_k)(v_k \otimes u_k))^{s_k} \cdot \prod_l (t_l \otimes (1 - t_l))^{d_l} .$$

Indeed, the quotient by elements of this form is Matsumoto's presentation for K_2, and by Tate's calculation, we have $K_2(\mathbb{Q}) \otimes \mathbb{Q} = 0$. Moreover, the proof amounts to a simple algorithm for producing such an expression. In terms of these decompositions, $h_{1,3}$ is bilinear given by

$$h_{1,3}(x, y) = \sum a_{i,j} x_i y_j$$

where

$$a_{i,j} = \sum_k s_k (\log u_k)(\log v_k) - \sum_l d_l (\mathrm{Li}\, t_l) .$$

Theorem (DC, Wewers)**.** The diagram

$$\begin{array}{ccc} X(\mathbb{Z}[S^{-1}]) & \longrightarrow & X(\mathbb{Z}_p) \\ \kappa \big\downarrow & & \big\downarrow \alpha \\ \mathbb{Q}^S \times \mathbb{Q}^S & \xrightarrow[h]{} & H(\mathbb{Q}_p) \end{array}$$

commutes.

Example

Let $S = \{2\}$. Then

$$h(x,y) = \begin{pmatrix} 1 & (\log 2)x & \frac{1}{2}(\log 2)^2 xy \\ & 1 & (\log 2)y \\ & & 1 \end{pmatrix}$$

so its image satisfies the equation

$$XY = 2Z$$

which pulls back to

$$\log(x)\log(y) = -2\,\mathrm{Li}(x)$$

on $X(\mathbb{Z}_p)$.

Now set $p = 3$. Then computations based on the work of Besser–de Jeu show that the roots are precisely $\{-1, 1/2, 2\}$.

The same holds for $p = 5, 7$.

For $p = 11$ you get more roots,

and after that the number of roots seems to increase quickly.

Proof

The determination of $h_{1,2}$, $h_{2,3}$, is a direct calculation in (abelian) p-adic Hodge theory, and was previously known. We turn our attention to $h_{1,3}$.

The category of mixed Tate motives over $\mathbb{Q}$ with $\mathbb{Q}$-coefficients unramified outside of S, is canonically

$$\mathbf{Rep}\, B,$$

where

$$B = \mathbb{G}_m \ltimes U,$$

where U is a prounipotent $\mathbb{Q}$-group. Its coordinate ring

$$A = \text{ regular functions on } U$$

has the structure of a graded Hopf algebra.

The p-adic de Rham realization of a mixed Tate motive has the structure of a mixed Tate filtered φ module.

The category of mixed Tate filtered φ modules has a similar structure;

we decorate with dR.

We construct a diagram like so.

$$
\begin{array}{ccccc}
X(\mathbb{Z}[S^{-1}]) & \longrightarrow & X(\mathbb{Z}_p) & & \\
\bar{\kappa}\downarrow & & \downarrow\bar{\alpha} & & \\
A_2 & \xrightarrow[\rho]{} & A_2^{\mathrm{dR}} & \xrightarrow{\Psi} & \mathbb{Q}_p \\
\nu\downarrow\cong & & & & \\
A_1 \otimes A_1 & & & & \\
\| & & & & \\
\mathbb{Q}^S \otimes \mathbb{Q}^S & & & &
\end{array}
$$

The theory of the unipotent fundamental group assigns to every point $(x, y) \in X(\mathbb{Z}[S^{-1}])$ a B-equivariant H-torsor. By the general theory of mixed Tate categories, such an object gives rise to an element $\bar{\kappa}(x) \in A_2$.

A similar description applies to $\bar{\alpha}$.

Commutativity of the square follows from Olsson's unipotent p-adic Hodge theory.

As for Ψ, let me just say vaguely that it comes from comparing the slope decomposition with the Hodge filtration.

Tate's decomposition into elements of the form

$$u \otimes v + v \otimes u \tag{D}$$

and elements of the form

$$t \otimes (1 - t) \tag{G}$$

takes place in the lower left (after possibly enlarging S). Elements of type G ("geometric elements") come from X, and explicit formulas for α obtained by Furusho deliver the answer

$$h_{1,3}(t \otimes r) = -\operatorname{Li}(t)\,.$$

On the other hand, elements of type D are in a certain sense *decomposable*, so the computation of their image reduces to the abelian case.