THE INDUCTION STEP IN THE WILDLY RAMIFIED HIGHER CLASS FIELD THEORY OF KERZ-SAITO IN THE ALGEBRAIC GEOMETRY LEARNING SEMINAR IN ESSEN FEBRUARY 2015

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Abstract. The main goal of this talk is to explain the role played by the cycle conductor in the wildly ramified higher class field theory of Kerz–Saito. Our main reference is Kerz-Saito [KS].

1. Statement of Theorem

1.1. Let X be normal and proper over a finite field k, let $C = \bigcup_\lambda C_\lambda$ be a union of locally principal curves. Assume $U = X \setminus C$ is smooth over k. If Z is an integral scheme with function field K and z is a closed point, we write K^h_z for the fraction field of the Henselization of Z at z. If Z is an integral curve in X not contained in C, we write

$$\psi_Z : Z^N \to Z$$

for the normalization and put

$$Z^N_\infty = \left\{ y \in Z^N \;\middle|\; \psi_Z(x) \in C \right\}$$

and

$$K(Z^N)^h_\infty = \prod_{y \in Z^N_\infty} K(Z^N)^h_y$$

The Wiesend class group is given by

$$W(U) = \operatorname{Coker}\left(\bigoplus_{Z \subset X} K(Z)^* \to \bigoplus_{Z \subset X} K(Z^N)^{h*}_\infty \oplus Z_0(U) \right).$$

Let $\mathcal{O}^h_{Z^N,\infty} = \prod_{y \in Z^N_\infty} \mathcal{O}^h_{Z^N,y}$. Let D be an effective Cartier divisor supported on C with ideal sheaf I_D. We let

$$\hat{F}^{(D)} W(X, C) \subset W(U)$$

denote the subgroup generated by the images of the groups

$$1 + I_D \mathcal{O}^h_{Z^N,\infty}$$

for all integral curves Z not contained in C. We define

$$C(X, D) = W(U)/\hat{F}^{(D)} W(X, C).$$

We define

$$C(U) = \lim_{\substack{\leftarrow \\ D}} C(X, D).$$

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1.2. Let K be a Henselian discrete valuation field of characteristic $p > 0$. For each $s \geq 1$ we have the isomorphism of Artin-Schreier-Witt theory

$$\delta_s : W_s(K)/(1 - Frob)W_s(K) \to H^1(K, \mathbb{Z}/p^s\mathbb{Z})$$

where W_s denotes the ring of Witt vectors of length s. We set

$$\operatorname{Fil}^{\log}_m W_s(K) = \big\{(a_{s-1}, \ldots, a_0) \in W_s(K) \;\big|\; p^i v(a_i) \geq -m\big\},$$
$$\operatorname{Fil}_m W_s(K) = \operatorname{Fil}^{\log}_{m-1} W_s(K) + Ver^{s-s'} \operatorname{Fil}^{\log}_m W_{s'}(K)$$

where

$$s' = \min\{s, \operatorname{ord}_p(m)\}.$$

We abbreviate $H^1(K) = H^1(K, \mathbb{Q}/\mathbb{Z})$. We set

$$\operatorname{Fil}_m H^1(K) = H^1(K)_{\text{prime to } p} \oplus \bigcup_{s \geq 1} \delta_s(\operatorname{Fil}_m W_s(K)).$$

1.3. We define

$$\operatorname{Fil}_D H^1(U) \subset H^1(U)$$

by the property

$$\chi|_{K(Z^N)^h_x} \in \operatorname{Fil}_{m_x(\psi_Z^* D)} H^1(K(Z^N)^h_x)$$

for all integral curves Z in X not contained in C and all points $x \in Z^N_\infty$. We define

$$\pi_1^{ab}(X, D) = \operatorname{Hom}(\operatorname{Fil}_D H^1(U), \mathbb{Q}/\mathbb{Z}).$$

1.4. We have

$$\hat{F}^{(D+C)} \subset \hat{F}^{(D)} \subset W(U)$$

from which exact sequences

$$0 \to \operatorname{gr}_D C(U) \to C(X, D + C) \to C(X, D) \to 0$$

or dually

$$0 \leftarrow \operatorname{gr}^{(D+C)} C(U)^\vee \leftarrow C(X, D + C)^\vee \leftarrow C(X, D)^\vee \leftarrow 0.$$

Roughly we should be thinking of "cycles modulo units congruent to 1 modulo D".

On the other hand we have

$$\operatorname{Fil}_D H^1(U) \to \operatorname{Fil}_{D+C} H^1(U) \to \operatorname{gr}_{D+C} H^1(U) \to 0.$$

We should think of $\operatorname{Fil}_D H^1(U)$ as "characters with ramification bounded by D.

1.5. We've seen that there's a unique map ρ_U compatible with the map sending a closed point x to Frobenius at x, which moreover induces maps $\rho_{X,D}$, as in the following diagram.

$$\begin{array}{ccccccccc}
Z_0(U) & \longrightarrow & C(U) & \longrightarrow \cdots \longrightarrow & C(X, D + C) & \longrightarrow & C(X, D) & \longrightarrow & \cdots \\
& \searrow & \downarrow{\scriptstyle \rho_U} & & \downarrow & & \downarrow{\scriptstyle \rho_{D,X}} & & \\
& & \pi_1^{ab}(U) & \longrightarrow \cdots \longrightarrow & \pi_1^{\mathrm{ab}}(X, D + C) & \longrightarrow & \pi_1^{ab}(X, D) & \longrightarrow & \cdots
\end{array}$$

Theorem. The map

$$\Psi_{X,D} : \operatorname{Fil}_D H^1(U) \to C(X, D)^\vee$$

induced by $\rho_{X,D}$ is iso.

1.6. We have

$$\pi_1^{\mathrm{ab}}(X, C) = \pi_1^{\mathrm{ab},t}(U).$$

Indeed, a character

$$\chi : \pi_1(U) \to \mathbb{Q}/\mathbb{Z}$$

is tame if its prime to p part is unramified. So it's enough to show that any character in the image of $\mathrm{Fil}_1 W_s(K)/1 - \mathit{Frob}$ extends to $H^1(\mathcal{O}_K, \mathbb{Z}/p^s)$. But one checks that

$$\mathrm{Fil}_1 = W_s(\mathcal{O}_K).$$

1.7. Injectivity follows from the Chebotarev density theorem. Roughly speaking, the base case

$$\mathrm{Fil}_C H^1(U) \twoheadrightarrow C(X, C)^\vee$$

follows from the tame case. For the induction we may then consider the following diagram,

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Fil}_D H^1 & \longrightarrow & C(X, D)^\vee \\
\downarrow & & \downarrow \\
\mathrm{Fil}_{D+C} H^1(U) & \longrightarrow & C(X, D + C)^\vee \\
\downarrow & & \downarrow \\
\mathrm{gr}_{D+C} H^1(U) & \xrightarrow[\mathrm{gr}_{D+C} \Psi_{X,D+C}]{} & \mathrm{gr}_{D+C} C(U)^\vee \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

and hope to show that $\mathrm{gr}_{D+C} \Psi_{X,D+C}$ is surjective by studying certain maps to an auxiliary space which may be regarded as developments on the Artin conductor. Before we can make this work, we have to make things quite a bit more complicated. One reason for this is that while we have a global cycle conductor, we only have a local Galois conductor. Another reason is that we will need the local Galois conductor to be surjective, which requires the residue field to be perfect.

2. Stament of Main Proposition

2.1. Let Y' be a normal scheme, and Y an open subscheme. For $y \in Y'$, we write $(Y')^h_y$ for the Heselization of the local scheme at y, and $K(Y')^h_y$ for its function field. For

$$\chi \in H^1(Y) := H^1(Y_{\text{ét}}, \mathbb{Q}/\mathbb{Z})$$

and D a divisor supported on $Y' \setminus Y$, we consider the *point-on-curve condition over* D

POC($Y \subset Y' \supset D$): for all integral curves Z on Y', and all $y \in Z^N$ mapping to the support of D, χ maps to

$$\mathrm{Fil}_{m_y(\psi_Z^* D)} H^1\big(K(Z^N)^h_x\big).$$

Note that we allow arbitrary ramification over points of $Y' \setminus Y$ not in the support of D. We compare the point-on-curve condition with the *height-one condition at generic points of* D

HO($Y \subset Y' \supset D$): for all generic points λ of $\mathrm{Supp}\, D$, χ maps to

$$\mathrm{Fil}_m H^1\big(K(Y')^h_\lambda\big).$$

As far as I understand, work of Matsuda [?] and Kato [?] on Swan conductors shows the following

Proposition. Suppose the support of D is regular. Then

$$\mathbf{POC}(Y \subset Y' \supset D) \qquad \text{if and only if} \qquad \mathbf{HO}(Y \subset Y' \supset D).$$

2.2. Let S be a proper smooth curve over a finite field k, $f : X \to S$ proper surjective with smooth generic fiber with X normal and of dimension 2, $\sigma : X \leftarrow S$ a section. Let C be a finite union of locally prinicipal integral curves C_λ with generic point λ. We assume $\sigma(S) \cap C$ doe not dominate S and that $U = X \setminus C$ is smooth over S. Finally, assume each dominant component

$$C_\lambda \to S$$

is an isomorphism.

2.3. Let $\eta = \operatorname{Spec} K$ denote the generic point of S and fix an algebraic closure $\bar\eta = \operatorname{Spec} \bar K$. We consider pairs $\Sigma = (T, \theta)$ where $T \to S$ is a smooth curve with function field contained in $\bar K$ and θ is an effective divisor on T.

Given Σ, we let U_Σ denote the pull back of U to $T \setminus \theta$, we let X_Σ denote the normalization of X_T, we let C_Σ denote the pullback of C to X_Σ, we let θ_Σ denote the pullback of θ to X_Σ.

We define a directed partial ordering by

$$\Sigma_1 \le \Sigma_2$$

if there's a factorization

$$T_2 \xrightarrow{g} T_1 \to S$$

and

$$g^*\theta_1 \le \theta_2.$$

We then have a map

$$U_{\Sigma_2} \to U_{\Sigma_1}$$

giving rise for each $m \ge 1$ to

$$\begin{array}{ccc} H^1(U_{\Sigma_2}) & \longleftarrow & H^1(U_{\Sigma_1}) \\ \uparrow & & \uparrow \\ \operatorname{Fil}_{mC_{\Sigma_2}+\theta_{\Sigma_2}} & \longleftarrow & \operatorname{Fil}_{mC_{\Sigma_1}+\theta_{\Sigma_1}} . \end{array}$$

2.4. Claim. We have isomorphisms

$$\begin{array}{ccc} H^1(U_{\bar\eta}) & \xleftarrow{\cong} & \varinjlim_{\Sigma} H^1(U_\Sigma) \\ \uparrow & & \uparrow \\ \operatorname{Fil}_m H^1(U_{\bar\eta}) & \xleftarrow[\cong]{h} & \varinjlim_{\Sigma} \operatorname{Fil}_{mC_\Sigma+\theta_\Sigma} H^1(U_\Sigma). \end{array}$$

Sketch of proof. The isomorphism at the top is a general fact about compatibility of étale cohomology with inverse limits of schemes. Regarding h, we have to establish existence of the factorization, and surjectivity. Note that throughout the proof of this claim, we may assume that all components of C_Σ dominate T by taking $\operatorname{Supp}\theta$ large enough. We start with existence.

For $\operatorname{Supp}\theta$ large enough, C_Σ is regular on $X_\Sigma \setminus \operatorname{Supp}\theta$. This puts us in the situation studied by Matsuda and Kato:

$$\mathbf{POC}(U_\Sigma \subset X_\Sigma \setminus \operatorname{Supp}\theta \supset C_\Sigma \setminus \operatorname{Supp}\theta) \quad \text{if and only if} \quad \mathbf{HO}(U_\Sigma \subset X_\Sigma \setminus \operatorname{Supp}\theta \supset C_\Sigma \setminus \operatorname{Supp}\theta).$$

Let $\bar{\lambda}$ be a closed point of $C_{\bar{\eta}}$ with image λ in X_Σ. Then we have a commutative diagram like so.

$$\begin{array}{ccc}
\mathrm{Fil}_m & \leftarrow\cdots & \mathrm{Fil}_m \\
\uparrow & & \uparrow \\
H^1\big(K(X_{\bar{\eta}})^h_{\bar{\lambda}}\big) & \xleftarrow[L]{} & H^1\big(K(X_\Sigma)^h_\lambda\big) \\
\uparrow & & \uparrow \\
H^1(U_{\bar{\eta}}) & \longleftarrow & H^1(U_\Sigma)
\end{array}$$

This gives us

$$\mathrm{Fil}_m H^1(U_{\bar{\eta}}) \leftarrow \mathrm{Fil}^{HOP}_{mC_\Sigma \setminus \theta_\Sigma}(U_\Sigma) = \mathrm{Fil}_{mC_\Sigma \setminus \theta_\Sigma}(U_\Sigma) \supset Fil_{mC_\Sigma + \theta_\Sigma}(U_\Sigma)$$

hence the factorization.

We turn to surjectivity. Starting with an arbitrary Σ, suppose $\chi \in H^1(U_\Sigma)$ maps to $\mathrm{Fil}_m H^1(U_{\bar{\eta}})$. The map L is strict with respect to the filtrations (directly from the definition in terms of Witt vetors), so that χ fulfills $\mathbf{HO}(U_\Sigma \subset X_\Sigma \supset C_\Sigma)$ hence also $\mathbf{HO}(U_\Sigma \subset X_\Sigma \setminus \theta_\Sigma \supset C_\Sigma \setminus \theta_\Sigma)$. For $\mathrm{Supp}\,\theta$ large enough, we then have $\mathbf{HO}(U_\Sigma \subset X_\Sigma \setminus \theta_\Sigma \supset C_\Sigma \setminus \theta_\Sigma)$ by Matsuda and Kato. Presumably there's a bound for possible ramification on points on curves, so for θ large enough, it follows that χ fulfills $\mathbf{HO}(U_\Sigma \subset X_\Sigma \supset C_\Sigma + \theta)$. ☐

2.5. Main Proposition. For each $m \geq 1$, the map

$$\Psi^{(m)}_{\bar{\eta}} : \mathrm{Fil}_m(U_{\bar{\eta}}) \xrightarrow[\cong]{h^{-1}} \varinjlim_\Sigma \mathrm{Fil}_{mC_\Sigma + \theta_\Sigma} H^1(U_\Sigma) \xrightarrow{\varinjlim_\Sigma \rho^\vee_{X_\Sigma, mC_\Sigma + \theta_\Sigma}} \varinjlim_\Sigma C(X_\Sigma, mC_\Sigma + \theta_\Sigma)^\vee$$

is surjective.

3. Review of key theorems

3.1. Let X be projective and smooth over $\mathbb{F}_q$, $C = \sum_\lambda C_\lambda$ the support of a Cartier divisor with complement U as above. Let $D = \sum_\lambda m_\lambda C_\lambda$ and assume $m_\lambda \geq 1$ for all λ. Define $\mathrm{gr}_D\, C(U)$ by

$$0 \to C(X, D)^\vee \to C(X, D + C)^\vee \to \mathrm{gr}_D\, C(U) \to 0.$$

Let K be a Henselian dvf of characteristic $p > 0$. Set

$$\mathrm{Fil}_m \Omega^1_K := \mathfrak{m}_K^{-m} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}.$$

Let C_λ denote a component of C with generic point λ and let $K(U)^h_\lambda$ denote the henselian local field of U at λ. We've constructed a divisor Θ not containing of components of C, a *local Galois conductor* rar and a *global cycle conductor* $\mathrm{cc}_{X,D}$, which form a commutative diagram like so.

$$\begin{array}{ccccc}
\mathrm{gr}_D H^1(U) & \longrightarrow & \mathrm{gr}_{m_\lambda} H^1(K(U)^h_\lambda) & \xrightarrow{\mathrm{rar}} & \mathrm{gr}_{m_\lambda} \Omega^1_{K(U)^h_\lambda} \\
\downarrow & & & & \downarrow \cong \\
\mathrm{gr}_D C(X)^\vee & \xrightarrow[\mathrm{cc}_{X,D}]{} & H^1(C, \Omega^1_X(D + \Theta)|_C) & \longrightarrow & \Omega^1_X(D) \otimes_{\mathcal{O}_X} K(C_\lambda)
\end{array}$$

Moreover, $\mathrm{cc}_{X,D}$ is injective.

3.2. It is also a fact that the local Galois conductor of a Heselian dvf with perfect residue field is surjective.

4. The induction step

4.1. There's an exact sequence

$$0 \to H^1(X_{\bar\eta}) \to H^1(U_{\bar\eta}) \to \bigoplus_\lambda H^1(K(U_{\bar\eta})^h_\lambda) \to H^2(X_{\bar\eta}, \mathbb{Q}/\mathbb{Z})$$

from which

$$\operatorname{gr}_m H^1(U_{\bar\eta}) = \bigoplus \operatorname{gr}_m H^1(K(U_{\bar\eta})^h_\lambda)$$

since $H^2(X_{\bar\eta}, \mathbb{Q}/\mathbb{Z})$ contains no p-torsion and $\operatorname{gr}_m H^1(K(U_{\bar\eta})^h_\lambda)$ is a p-group.

Since each $K(U_{\bar\eta})^h_\lambda$ has perfect residue field $\overline{K(C_\lambda)}$, the local Galois conductors

$$\operatorname{rar} : \operatorname{gr}_m H^1(K(U_{\bar\eta})^h_\lambda) \to \Omega^1_{X_{\bar\eta}}(mC) \otimes \overline{K(C_\lambda)}$$

are surjective.

4.2. The X_Σ need not be nonsingular. However, for θ sufficiently large, desingularization does not affect the Chow group. So we have the global cycle conductor at our disposal:

$$\operatorname{gr}_{mC_\Sigma + \theta_\Sigma} C(X_\Sigma)^\vee \hookrightarrow H^0\big(C_\Sigma \cup \theta_\Sigma, \Omega^1_X(mC_\Sigma + \theta_\Sigma + \Theta)|_{C_\Sigma \cup \theta_\sigma}\big).$$

The latter maps to

$$\bigoplus_\lambda \Omega^1_{X_\Sigma}(mC_\Sigma) \otimes K(C_\lambda).$$

The sum runs over generic points λ of C which dominate S. This map becomes injective in the limit over θ. The resulting triangle

$$\begin{array}{ccc} \operatorname{gr}_m H^1(U_{\bar\eta}) & \longrightarrow & \operatorname{gr}_m \varinjlim C(X_\Sigma, mC_\Sigma + \theta_\sigma)^\vee \\ & \searrow \quad \swarrow & \\ & \bigoplus_\lambda \Omega^1_{X_{\bar\eta}}(mC_{\bar\eta}) \otimes \overline{K(C_\lambda)} & \end{array}$$

commutes, showing that the map of graded pieces is iso. This completes the induction step.

References

[KS] Kerz and Saito. Chow group of 0-cycles with modulus and higher dimensional class field theory. Preprint. arXiv:1304.4400.

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