DIFFERENTIAL TANNAKIAN FORMALISM

MOSHE KAMENSKY

The purpose of this note is to define differential tensor categories, and to give a model theoretic proof of the basic theorem, corresponding such categories, endowed with a suitably defined fibre functor, with linear differential algebraic groups. This result was proved by algebraic methods in [Ovc07a] and [Ovc07b].

1. Derivatives of categories

All categories are abelian and all functors are additive unless stated otherwise. We also assume that in a tensor category \( (C, \otimes) \), the functor \( \otimes \) is exact; this is automatic if \( C \) is rigid (see [DMOS82].)

1.1. Definition. Let \( C \) be an abelian category. The derivative \( D(C) \) of \( C \) is defined as follows: The objects are exact sequences \( 0 \rightarrow X_0 \rightarrow X_1 \rightarrow X_0 \rightarrow 0 \) of \( C \), and the morphisms from such an object are morphisms of exact sequences whose two \( X \) parts coincide.

The category \( D(C) \) is again abelian. An exact functor \( F : C_1 \rightarrow C_2 \) gives rise to an induced (exact) functor \( D(F) : D(C_1) \rightarrow D(C_2) \). We denote by \( \Pi_i (i = 0, 1) \) the functors from \( D(C) \) to \( C \) assigning \( X_i \) to \( 0 \rightarrow X_0 \rightarrow X_1 \rightarrow X_0 \rightarrow 0 \) (thus there is an exact sequence \( 0 \rightarrow \Pi_0 \rightarrow \Pi_1 \rightarrow \Pi_0 \rightarrow 0 \).) \( \Pi_i(X) \) is also abbreviated as \( X_i \), and \( X \) is said to be over \( X_0 \) (and similarly for morphisms.)

1.2. Let \( A \) and \( B \) be two objects over \( X \). Their Yoneda sum \( A \star B \) is a new object over \( X \), defined as follows: the combined map \( X \times X \rightarrow A_1 \times B_1 \) factors through \( A \times X \times B_1 \), and together with the map \( \frac{1}{X} \rightarrow X \times X \) gives rise to a map \( f : X \rightarrow A_1 \times X \times B_1 \). Let \( W \) be the co-kernel of this map. The map \( f \) composed with the projection from \( A_1 \times X \times B_1 \) to \( X \) is 0, so we obtain an induced map \( p : W \rightarrow X \). The diagonal inclusion \( \Delta \) of \( X \) in \( W \) together with \( p \) give rise to an exact sequence \( 0 \rightarrow X \xrightarrow{\Delta} W \xrightarrow{p} X \rightarrow 0 \), which is the required object.

For any object \( A \) of \( D(C) \), we denote by \( T(A) \) the object obtained by negating all arrows that appear in \( A \).

1.3. Let \( (C, \otimes, \phi, \psi) \) be a tensor category. An object \( X_0 \) of \( C \) gives rise to a functor from \( D(C) \) to itself, by tensoring the exact sequence pointwise. Since we assumed \( \otimes \) to be exact, this functor commutes with Yoneda sums: \( (A \star B) \otimes X_0 \) is canonically isomorphic with \( (A \otimes X_0) \star (B \otimes X_0) \). Also, \( T(A) \otimes X_0 \) is isomorphic to \( T(A \otimes X_0) \).

We endow \( D(C) \) with a tensor structure. The tensor product of the two \( D(C) \) objects \( 0 \rightarrow A_0 \xrightarrow{i_0} A_1 \xrightarrow{\pi_0} A_0 \rightarrow 0 \) and \( 0 \rightarrow B_0 \xrightarrow{i_0} B_1 \xrightarrow{\pi_0} B_0 \rightarrow 0 \) is defined as follows: After tensoring the first with \( B_0 \) and the second with \( A_0 \), we obtain two objects over \( A_0 \otimes B_0 \). We now take their Yoneda sum.

We shall make use of the following exact sequence.
1.4. **Lemma.** for any two objects $A$ and $B$ of $\mathbb{D}(C)$, there is an exact sequence

$$0 \to (A \otimes T(B))_{1} \xrightarrow{i} A_{1} \otimes B_{1} \xrightarrow{\pi} (A \otimes B)_{1} \to 0 \quad (1)$$

where $\pi$ is the quotient of the map obtained from the maps $\pi_A \otimes 1$ and $1 \otimes \pi_B$, and $i$ is the restriction of the map obtained from the maps $i_A \otimes 1$ and $-1 \otimes i_B$.

**Proof.** Exactness in the middle follows directly from the definitions. We prove that $\pi$ is surjective, the injectivity of $i$ being similar. We shall use the Mitchell embedding theorem (cf. [Fre64]), which reduces the question to the case of abelian groups.

We in fact prove that already the map $A_1 \otimes B_1 \xrightarrow{\pi} A_0 \otimes B_1 \times A_0 \otimes B_0 \times A_1 \otimes B_0 =: U$

is surjective. Let $y$ be an element of $U$, and let $y_1$ and $y_2$ be its two projections to the components of $U$. Since the map $A_1 \otimes B_1 \xrightarrow{\pi_A \otimes 1} A_0 \otimes B_1$ is surjective, $y_1$ can be lifted to an element $\hat{y}_1$ of $A_1 \otimes B_1$. We have that $$(\pi_A \otimes 1)((1 \otimes \pi_B)(\hat{y}_1)) = (1 \otimes \pi_B)((\pi_A \otimes 1)(\hat{y}_1)) = (1 \otimes \pi_B)(y_1) = (\pi_A \otimes 1)(y_2)$$

Let $z = (1 \otimes \pi_B)(\hat{y}_1) - y_2$. Since $z$ is killed by $\pi_A \otimes 1$, it comes from an element, also $z$, of $A_0 \otimes B_0$. Let $\tilde{z}$ be a lifting of $z$ to $A_0 \otimes B_1$, and denote by $\tilde{y}$ also its image in $A_1 \otimes B_1$ under the inclusion $i_A \otimes 1$. Then $\hat{y}_1 - \tilde{z}$ is a lifting of $y$. \qed

1.5. Let $A, B, C$ be three objects of $\mathbb{D}(C)$. The associativity constraint $\phi_0$ of $C$

gives rise to an isomorphism of $(A \otimes B) \otimes C$ with the quotient of

$$A_1 \otimes B_0 \otimes C_0 \times A_0 \otimes B_0 \otimes C_0 \times A_0 \otimes B_0 \otimes C_0 \times A_0 \otimes B_0 \otimes C_1$$

that identifies the three natural inclusions of $A_0 \otimes B_0 \otimes C_0$, and similarly for $A \otimes (B \otimes C)$. We thus get an associativity constraint $\phi$ on $\mathbb{D}(C)$, over $\phi_0$.

Likewise, the commutativity constraint $\psi_0$ induces a commutativity constraint $\psi$ on $\mathbb{D}(C)$ over $\psi_0$.

1.6. **Proposition.** The data $(\mathbb{D}(C), \otimes, \phi, \psi)$ as defined above forms a tensor category, and $\Pi_0$ is a tensor functor. It is rigid if $C$ is rigid.

**Proof.** We define the additional data. Verification of the axioms reduces, as in lemma [4], to the case of abelian groups, where it is easy.

Let $u : 1_0 \to 1_0 \otimes 1_0$ be an identity object of $C$. We set $1 = 0 \to 1_0 \to 1_0 \otimes 1_0 \to 1_0$. For any object $A$ of $\mathbb{D}(C)$, $1 \otimes A$ is identified via $u$ with $0 \to A_0 \to (A_1 \times A_0 \times A_0) \to A_0 \to 0$. This is canonically isomorphic (over $C$) to $A$, and so $1$ acquires a structure of an identity object.

Assume that $C$ is rigid. For an object $A$ of $\mathbb{D}(C)$, we set $\tilde{A}$ to be the dual exact sequence $0 \to \tilde{A}_0 \xrightarrow{i_A} \tilde{A}_1 \xrightarrow{\pi_A} \tilde{A}_0 \to 0$. We define an evaluation map $A \otimes \tilde{A} \to 1$ as follows: We need to define two maps from $A_0 \otimes \tilde{A}_1 \times A_1 \times \tilde{A}_0$ to $1_0$, that agree on the two inclusions of $A_0 \otimes \tilde{A}_0$, and such that the resulting map restricts to the evaluation on $A_0 \otimes \tilde{A}_0$.

To construct the first map, we consider the exact sequence $1$, for $B = \tilde{A}$. We claim that the evaluation map on $A_1 \otimes \tilde{A}_1$ restricts to 0 when composed with $i$. To prove this, it is enough to show that the pair of maps obtained from $ev_{A_1}$, by composition with $i_A \otimes 1$ and $-1 \otimes i_{\tilde{A}}$ comes from a map $A_0 \otimes \tilde{A}_0 \to 1_0$. However, under the adjunction, this pair of maps corresponds to $(i_A, -\pi_A)$, and so comes from the identity map on $A_0$. It follows that $ev_{A_1}$ induces a map on $(A \otimes \tilde{A})_1$, \ldots
which is the required map. The second map is obtained by projecting to \( A_0 \otimes \hat{A}_0 \), and using the evaluation map on \( A_0 \). By definition, this second map commutes with the projections to \( A_0 \otimes \hat{A}_0 \) and the second coordinate of \( 1 \), restricting to the evaluation on \( A_0 \). To prove that the first map restricts to the evaluation as well, we note that there is a commutative diagram

\[
\begin{array}{ccc}
(A \otimes \hat{A})_1 & \xrightarrow{i} & A_1 \otimes \hat{A}_1 \\
\pi_{A \otimes A} & & \pi \\
A_0 \otimes \hat{A}_0 & \xrightarrow{i_{A \otimes A}} & (A \otimes \hat{A})_1 
\end{array}
\]

where \( i \) is the (restriction of the) map obtained from the two maps \( i_A \otimes 1 \) and \( 1 \otimes i_A \).

Since \( \pi_{A \otimes A} \) is surjective, it is therefore enough to prove that the maps \( ev_{A_1} \circ i \) and \( ev_{A_0} \circ \pi_A \) coincide. This is indeed the case, since they both correspond to the inclusion of \( A_0 \) in \( A_1 \).

\[\Box\]

2. Differential tensor categories

2.1. **Definition.** A differential structure on a tensor category \( C \) is a tensor functor from \( C \) to \( \mathbb{D}(C) \) which is a section of \( \Pi_0 \). If \( D_1 \) and \( D_2 \) are two differential structures on \( C \), a morphism from \( D_1 \) to \( D_2 \) is a morphism of tensor functors that induces the identity morphism under \( \Pi_0 \). A differential tensor category is a tensor category together with a differential structure.

Let \( D \) be a differential structure on \( C \). Since \( D \) is a section of \( \Pi_0 \), it is determined by \( \partial = \Pi_1 \circ D \). In other words, on the abelian level, it is given by a functor \( \partial : C \to C \), together with an exact sequence \( 0 \to \text{Id} \to \partial \to \text{Id} \to 0 \). However, this description does not include the tensor structure. We also note that \( D \) (and \( \partial \)) is necessarily exact.

2.2. Let \((C, D)\) be a differential tensor category, let \( \partial = \Pi_1 \circ D \), and let \( A = \text{End}(1) \). Recall that for any object \( X \), \( \text{End}(X) \) is an \( A \)-algebra. The functor \( \partial \) defines another ring homomorphism \( \partial_1 : A \to \text{End}(\partial(1)) \). Given \( a \in A \), the morphism \( \partial_1(a) - a \) in \( \text{End}(\partial(1)) \) restricts to \( 0 \) on \( 1 \), and thus induces a morphism from \( 1 \) to \( \partial(1) \). Similarly, its composition with the projection \( \partial(1) \to 1 \) is \( 0 \), so it factors through \( 1 \). We thus get a new element \( a' \) of \( A \).

**Claim.** The map \( a \mapsto a' \) is a derivation on \( A \).

**Proof.** We need to show that given elements \( a, b \in A \), the maps \( \partial(ab) - ab \) and \( (\partial(a) - a)b + a(\partial(b) - b) \) coincide on \( 1 \). This follows from the formula \( \partial(ab) - ab = \partial(a)(\partial(b) - b) + (\partial(a) - a)b \), together with the fact that \( \partial(a)(\partial(b) - b) \) induces \( a(\partial(b) - b) \) on \( 1 \). \(\Box\)

2.3. **Example.** Let \( C \) be the tensor category \( \mathcal{Vec}_k \) of finite dimensional vector spaces over a field \( k \). Given a derivative \( \partial \) on \( k \), we construct a differential structure on \( C \) as follows: For a vector space \( X \), define \( d(X) = D \otimes X \), where \( D \) is the vector space with basis \( 1, \partial \), and \( \otimes \) is the tensor product with respect to the right vector space structure on \( D \), given by \( 1 \cdot a = a \cdot 1 \) and \( \partial \cdot a = a' \cdot 1 + a \cdot \partial \). The exact sequence \( D(X) \) is defined by \( x \mapsto 1 \otimes x, 1 \otimes x \mapsto 0 \) and \( \partial \otimes x \mapsto x \), for any \( x \in X \). If \( T : X \to Y \) is a linear map, \( d(T) = 1 \otimes T \). We shall write \( x \) for \( 1 \otimes x \) and \( \partial x \) for
∂⊗x. The structure of a tensor functor is obtained by sending ∂(x ⊗ y) ∈ d(X ⊗ Y) to the image of ∂(x) ⊗ y ⊕ x ⊗ ∂(y) in (D(X) ⊗ D(Y))₁.

2.4. Claim. The constructions in 2.3 and in 2.2 give a bijective correspondence between derivatives on k and isomorphism classes of differential structures on Vec_k.

Proof. If D₁ and D₂ are two differential structures, then D₁(1₀) and D₂(1₀) are both identity objects, and are therefore canonically isomorphic to the same object 1. If D₁ and D₂ are isomorphic, then the maps d₁ : End(1₀) → End(1) are conjugate, and therefore equal, since End(1) is commutative.

It is clear from the definition that the derivative on k obtained from the differential structure associated with a derivative is the original one. Conversely, if D₁ and D₂ are two differential structures that give the same derivative on k, then we may identify D₁(1₀) and D₂(1₀). Under this identification, we get that the maps d₁ are the same. But the functors D₁ are determined by d₁. □

2.5. We now come to the definition of functors between differential tensor categories. For simplicity, we shall only define (and use) exact such functors.

Let ω : C → D be an exact functor between abelian categories. There is an induced functor D(ω) : D(C) → D(D), given by applying ω to each term. If C and D are tensor categories, the structure of a tensor functor on ω gives rise to a similar structure on D(ω) (again, since ω is exact.) If t : ω₁ → ω₂ is a (tensor) morphism of functors, we likewise get an induced morphism D(t) : D(ω₁) → D(ω₂).

2.6. Definition. Let (C₁, D₁) and (C₂, D₂) be two differential tensor categories. A differential tensor functor from C₁ to C₂ is an exact tensor functor ω from C₁ to C₂, together with an isomorphism of tensor functors r : D(ω) ◦ D₁ → D₂ ◦ ω.

A morphism between two such differential tensor functors (ω₁, r₁) and (ω₂, r₂) is a morphism t between them as tensor functors such that the following diagram (of tensor functors and tensor maps between them) commutes:

\[
\begin{array}{ccc}
D(ω₁) ◦ D₁ & \overset{r₁}{\longrightarrow} & D₂ ◦ ω₁ \\
\downarrow D(t) ◦ D₁ & & \downarrow D₂ ◦ t \\
D(ω₂) ◦ D₁ & \overset{r₂}{\longrightarrow} & D₂ ◦ ω₂
\end{array}
\] (2)

where D₂ ◦ t is the map from D₂ ◦ ω₁ to D₂ ◦ ω₂ obtained by applying D₂ to t “pointwise”.

Given a differential tensor functor ω, we denote by Aut^θ(ω) the group of automorphisms of ω.

If C is a differential tensor category, and k = End(1) is a field, a k-linear differential tensor functor into Vec_k (with the induced differential structure) is called a (differential) fibre functor. Given such a functor ω, we denote by G_ω the functor from differential k-algebras to groups assigning to an algebra A the group Aut^θ(A ⊗ ω).

2.7. Given a k vector space V, the map d : V → D⊗V given by v ↦ ∂v is a derivation, in the sense that d(av) = a′v + ad(v) (where V is identified with its image in D⊗V.) It is universal for this property: any pair (i, d) : V → W, where i is linear, and d is a derivation with respect to i factors through it.

Therefore, a fibre functor on (C, D) is a fibre functor ω in the sense of tensor categories, together with a functorial derivation dₓ : ω(X) → ω(∂X) (where
\[ \partial X = D(X)_1 \]
satisfying the Leibniz rule with respect to the tensor product (and additional conditions.)

Similarly, a differential automorphism of \( \omega \) is an automorphism \( t \) of \( \omega \) as a tensor functor, with the additional condition that for any object \( X \), the diagram

\[
\begin{align*}
\omega(X) & \xrightarrow{d_X} \omega(\partial X) \\
\downarrow t_X & \quad \downarrow t_{\partial X} \\
\omega(X) & \xrightarrow{d_X} \omega(\partial X)
\end{align*}
\]  

(3) \{eq:fibauto\}

commutes. Thus the condition (2) really is about preservation of the differentiation.

2.8. Example. Let \( G \) be a linear differential algebraic group over a differential field \( k \). A representation of \( G \) is given by a finite dimensional vector space \( V \) over \( k \), together with a morphism \( G \to GL(V) \). A map of representations is a linear transformation that gives a map of group representations for each differential \( k \)-algebra. The category of all such representations is denoted \( \text{Rep}_G \).

We endow \( \text{Rep}_G \) with a differential structure in the same way as for vector spaces. If \( V \) is a representation of \( G \), assigning \( gv \) to \( (g,v) \), then the action of \( G \) on \( D\hat{\otimes}V \) is given by \( (g,x\hat{\otimes}v) \mapsto x\hat{\otimes}gv \). With this differential structure, the forgetful functor \( \omega \) into \( \text{Vec}_k \) has an obvious structure of a differential tensor functor.

A differential automorphism \( t \) of \( \omega \) is given by a collection of vector space automorphisms \( t_V \), for any representation \( V \) of \( G \). The commutativity condition (3) above translates to the condition that \( t_{D\hat{\otimes}V} = 1\hat{\otimes}t_V \).

In particular, given a differential \( k \)-algebra \( A \), and \( g \in G(A) \), action by \( g \) gives an automorphism of \( A \otimes \omega \) as a differential tensor functor, since the action of \( g \) on \( D\hat{\otimes}V \) is deduced from its action on \( V \). Thus we get a map \( G \to G_\omega \). We shall prove in 3.10 that the map is an isomorphism.

2.9. Remark. By a linear differential algebraic group, we mean a differential algebraic group which is represented by a differential Hopf algebra. A differential algebraic group which is affine as a differential algebraic variety need not be linear in this sense, since a morphism of affine differential varieties need not correspond to a map of differential algebras. Any linear differential algebraic group has a faithful representation. All these results appear in [Cas72], along with an example of an affine non-linear group. In [Cas75] it is shown that any representation of a linear group (and more generally, any morphism of linear groups) does correspond to a map of differential algebras.

2.10. Example. Let \( G_m \) be the (differential) multiplicative group, and let \( \hat{G}_m \) be the multiplicative group of the constants (thus, as differential varieties, \( G_m \) is given by the equation \( xy = 1 \), and \( \hat{G}_m \) is the subvariety given by \( x' = 0 \).) There is a differential algebraic group homomorphism \( dlog \) from \( G_m \) to \( G_a \), sending \( x \) to \( x'/x \), and \( x \mapsto x' \) is a differential algebraic group endomorphism of \( G_a \). Let \( V \) be the standard 2-dimensional algebraic representation of \( G_a \) (identifying \( G_a \) with the maximal unipotent group of \( GL_2 \).) Using \( dlog \) and the derivative, we thus get for any \( i \geq 0 \) a 2-dimensional irreducible representation \( V_i \) of \( G_m \), which are all unrelated in terms of the tensor structure (and unrelated with the non-trivial 1-dimensional algebraic representations of \( G_m \).)
However, if $X$ is the $G_m$ representation corresponding to the identity map on $G_m$, an easy calculation shows that $V_0$ is isomorphic to $\partial X \otimes X$. Similarly, $V_i$ is a quotient of $\partial V_{i-1}$.

The inclusion of $G_m$ in $G_m$ gives a functor from $\text{Rep}_{G_m}$ to $\text{Rep}_{G_m}$. But in $\text{Rep}_{G_m}$, $V_0$ is isomorphic to $1 \otimes 1$ (and $\partial X$ to $X \otimes X$)

3. Model theory of differential fibre functors

3.1. Throughout this section, we work with a fixed differential rigid tensor category $(\mathcal{C}, \otimes, \phi, \psi, D)$, with $\mathbb{k} = \text{End}(1)$ a field. We view $\mathbb{k}$ as a differential field, with the differential structure induced from $D$, as in 2.2. We set $\partial = \Pi_1 \circ D$, and denote by $i$ and $p$ the maps in the exact sequence $0 \rightarrow I \partial \rightarrow \partial \rightarrow I \rightarrow 0$.

3.2. The theory associated with a fibre functor. We consider the following theory $T_{\mathcal{C}}$:

1. For any object $X$ of $\mathcal{C}$, $T$ has a sort $V_X$. The sort $V_1$ is denoted by also by $L$.
2. $L$ is a differentially closed field, with constants for the elements of $\mathbb{k}$ (with the prescribed differential field structure.) Every $V_X$ is a vector space over $L$, of dimension $rk(X)$.
3. For every morphism of $f : X \rightarrow Y$ of $\mathcal{C}$, there is a corresponding function symbol $V_f : V_X \rightarrow V_Y$. The theory says that all these functions are linear, and reflects the abelian category structure.
4. For every two objects $X$ and $Y$ there is a symbol for a bilinear map $b_{X,Y} : V_X \times V_Y \rightarrow V_{X \otimes Y}$, that induces an isomorphism (in any model) of $V_X \otimes V_Y$ with $V_{X \otimes Y}$. The associativity and commutativity constraints commute with their usual counterpart on sets: $V_{f_{X,Y}} \circ b_{X,Y} = b_{Y,X} \circ s$ (where $s$ is the map $(a, b) \mapsto (b, a)$), and similarly for associativity.
5. For every object $X$, there is a function symbol $d_X : V_X \rightarrow V_{\partial(X)}$. This function is a derivation, in the sense that for any $a \in L$ and $v \in V_X$,

$$d_X(av) = a'V_{i,a}(v) + ad_X(v)$$

The theory furthermore says $d_X$ identifies $V_{\partial(X)}$ with $\partial \otimes V_X$ (in any model), in the sense of 2.7 (explicitly, it says that $V_{\partial X} \circ d_X$ is the identity map.)
6. The maps $d$ and $b$ are compatible with the structure of tensor functor of $D$;

given objects $X$ and $Y$ of $\mathcal{C}$, let $c_{X,Y} : \partial(X \otimes Y) \rightarrow (\partial(X) \otimes \partial(Y))_1$ be the isomorphism supplied with $D$. Then we require that $V_{c_{X,Y}} \circ d_{X \otimes Y} \circ b_{X,Y}$ coincides with $b_{\partial(X),Y} \circ d_X \times 1 + b_{X,\partial(Y)} \circ 1 \times d_Y$.

3.3. Let $\omega$ be a differential fibre functor on $\mathcal{C}$, and let $K = M_1$ be a differentially closed field containing $k$. We expand $M_1$ to a model $M$ of $T_{\mathcal{C}}$ as follows: For any object $X$, $M_X = V_X(M)$ is $\omega(X) \otimes_k K$. For any morphism $f$, $f_M = V_f M$ is the map $\omega(f) \otimes 1$ (this satisfies 3.2.3 since tensoring by $K$ is an exact functor.)

The tensor functor structure on $\omega$ includes, for any objects $X, Y$, an isomorphism $c_{X,Y} : \omega(X) \otimes \omega(Y) \rightarrow \omega(X \otimes Y)$. The maps $b_{X,Y,M}$ are obtained by composing $c_{X,Y} \otimes 1$ with the bilinear map from $M_X \times M_Y$ to $M_X \otimes M_Y$. The axioms of a tensor functor ensure that 3.2.4 is satisfied.

Finally, the differential structure of $\omega$ gives (as in 2.7) a universal derivation $\omega(X) \rightarrow \omega(\partial(X))$, which extends uniquely to a (universal) derivation $(d_X)_M$ on $M_X$. This concludes the construction of $M$. 

3.4. The model $M_\omega$ just constructed contains, in a natural way, the subset of elements coming from the fibre functor $\omega$. This set, which will also be denoted by $\omega$ is definably closed: If $\tau$ is any automorphism of $K$ over $k$, it extends to an automorphism of $M$ by acting on the $K$ part of each sort. Since $k$ is definably closed in $K$, the set of elements fixed by all such automorphisms is precisely $\omega$. In particular, we get the following corollary:

**Corollary.** Assume that $C$ has a differential fibre functor. Then $T_C$ is consistent, and $\text{dcl}(0) \cap L = k$.

3.5. **Internality.** Each sort of $T_C$, being a finite dimensional vector space over $L$, is internal to $L$. Furthermore, if $B$ is a basis for some $V_X$, then $B \cup d_X(B)$ is a basis for $V_{\partial X}$. Therefore, if $C$ is generated as a differential tensor category by one object (in the sense that the objects $\partial^i X$ generate $C$ as a tensor category), then all of the sorts are internal using the same definable set: There is a definable set $U$ (namely, the set of bases of $V_X$, where $X$ is a generator), such that, for any sort $V_Y$ there is a definable family $g : V_Y \times U \to L^m$ of injective maps from $V_Y$ to some $L^m$, parametrised by $U$.

In these circumstances, the basic theory of internality gives rise to a definable automorphism group $G$ of $T_C$ over $L$, acting freely on $U$. If $A$ is a definably closed set such that $U(A)$ is not empty, then $G(A)$ is canonically identified (together with the action on $U$) with the group of automorphisms of $A$ over $L(A)$. If $C$ is not finitely generated, the automorphism group is the pro-definable group represented by the system of such group, for a presentation of the category as a union.

3.6. Let $\omega$ be a fibre functor, and recall (2.6) that $G_\omega$ denotes the group functor $A \mapsto \text{Aut}^\partial (A \otimes \omega)$ on the category of differential $k$-algebras. Let $G$ be the (pro-) definable group associated with $T_C$. If $A$ is a differential field extension of $k$, $A \otimes \omega$ can be considered as a subset of a model of $T_C$, which is definably closed, by the same argument as in 3.4. Therefore, $G(A \otimes \omega)$ is a well defined group, identified with the automorphisms of $A \otimes \omega$ over $A$ (as $T_C$ structures.) Since any automorphism in this sense is an automorphism of $A \otimes \omega$ as a fibre functor, we get a map $G(- \otimes \omega) \to G_\omega$ (restricted to fields), which is obviously injective. On the other hand, given an element of $G_\omega(A)$ we may extend it to a model by tensoring with the identity (on a differentially closed field extending $A$.) Therefore, the map is an isomorphism.

Since all our information is given in terms of definably closed sets, namely differential fields, the following lemma is useful.

3.7. **Lemma.** Let $G_1$ and $G_2$ be two definable groups in a stable theory $T$, and let $f$ be an homomorphism between the group functors they induce on definably closed sets. Then $f$ comes from a definable homomorphism.

We note that in the event that the $G_i$ are differential algebraic groups in $DCF$, $f$ is automatically an algebraic morphism.

**Proof.** The graph of $f$ is a sub-sheaf of $G_1 \times G_2$, and is therefore a union of types. Since it is the graph of a function, the (preimage of) the generic types of $G_1$ must be included in this set of types. But since $f$ is a group homomorphism, $f$ is determined by its values on these generic types. 

\[\text{TODO: explain this a bit more}\]
3.8. Let $H$ be a differential algebraic group over $k$, let $\mathcal{C} = \text{Rep}_H$ (example 2.8), and let $\omega$ be the forgetful functor. We consider $H$ as a functor on differential field extensions of $k$. The action of $H$ on its representations gives rise to a map $H \to G_\omega = G(- \otimes \omega)$, as in 2.8. By 3.7 this map is definable. We would like to prove the following theorem:

**Theorem.** The natural map from $H$ to $G_\omega$ is surjective.

Let $\tilde{H}$ be the image. Galois theory implies, that to prove the theorem, it is enough to prove that every element of $dcl(\omega)$ fixed by $\tilde{H}$ is definable over 0. However, here $dcl$ and “definable” include imaginaries. Therefore, to use this method, it is essential to describe the imaginaries in $T_{\mathcal{C}}$. The following is an analogue of [Hru06, Proposition 4.2].

3.9. **Proposition.** Let $T$ be a theory with sorts $L$ and $(V_X)_X$, where the restriction of $T$ to $L$ is DCF, $L$ is stably embedded, each $V_X$ is a finite-dimensional vector space over $L$, and the $V_X$ are closed under tensor products, duals and derivations, in the sense of [3.2(4), 3.2(5) and 3.2(6)]. Then $T$ eliminates imaginaries to the level of projective spaces.

**Proof.** We need to show that any definable set $S$ over parameters can be defined with a canonical parameter. Since, by assumption, no new structure is induced on $L$, and any set is internal to $L$, every such set is Kolchin constructible. By Noetherian induction, it is enough to consider $S$ Kolchin closed.

We note that if $U$ is a definable subspace of some $V$, and $V$ carries a differential data, then the restriction of the differential data to $U$ gives a differential data for $U$, and similarly for quotients. We denote by $d_U : U \to \partial(U)$ the universal derivation.

A Kolchin closed set is given by a finite number of differential polynomial equations, i.e., by a finite set of elements of structure algebra of some $U = V_X$. This algebra is the symmetric algebra, $S(U^*)$, on the vector space $U^* = k[\partial] \otimes U$, where $k[\partial]$ is the algebra of differential operators. The vector space $U^*$ has a natural filtration, $U^{(i)}$, by the degree of the differential operator. We claim, by induction, that $U^{(i)}$ (with its evaluation map on $U$) is definable in $T$. It will follow that the algebra of functions is defined as a differential algebra, since multiplication is given by the $b_{X,Y}$, and the derivation by the $d_X$.

More precisely, we shall construct by induction $U^{(i)}$, the exact sequence $0 \to U^{(0)} \to U^{(i)} \to U^{(i-1)} \to 0$, and a map $t_i$ from $U^{(i)}$ to $\partial(U^{(i-1)})$. For $i = 0$, $U^{(0)}$ is the dual $\hat{U}$ of $U$, so setting $U^{(-1)} = 0$ we are done.

Assuming $U^{(i)}$, with the auxiliary data, is definable, we consider two maps from $\partial(U^{(i)})$ to $\partial(U^{(i-1)})$. The first, $p_i$, is defined by composing the map $p_i$ from $\partial(U^{(i)})$ to $U^{(i)}$, given by the differential structure, with $t_i$. The second, $q_i$, is obtained by applying the functor $\partial$ to the exact sequence for $U^{(i)}$ (in other words, the projection from $U^{(i)}$ to $U^{(i-1)}$, composed with the universal derivation on $U^{(i-1)}$, gives a derivation on $U^{(i)}$, and therefore a linear map on $\partial(U^{(i)})$). We set $U^{(i+1)} = \text{Ker}(p_i - q_i)$, $t_{i+1}$ the inclusion map, and the exact sequence is obtained from restriction of the differential structure on $\partial(U^{(i)})$. Also, since both $p_i$ and $q_i$ send $U^{(i)}$ to 0, we get an inclusion map $U^{(i)} \to U^{(i+1)}$. The evaluation map $\epsilon_{i+1} : U^{(i+1)} \times U \to L$ is defined inductively as follows: if $u \in U$, the map $d \mapsto \epsilon_i(d,u)$ is a derivation on $U^{(i)}$, and so defines, together with the inclusion of $U^{(0)}$ in $U^{(i)}$, a linear map from
∂(U(i)) to L. \(e_{i+1}(-, u)\) is the restriction of this map to \(U^{(i+1)}\). Finally, \(e_0\) is the usual evaluation map.

The rest of the proof is the same as in [Hru06], namely, the Kolchin closed set \(S\) is determined by the finite dimensional linear space spanned by the defining equations, and this space is an elements of some Grassmanian, which is, in turn, a closed subset of some projective space. □

3.10. **Proof of 3.8** As explained in 3.8, we should prove that any element of \(dcl(\omega)\) fixed by \(H\) is 0-definable, where \(dcl\) is computed in the theory \(T_C\) expanded by sorts for the projective spaces. Thus we need to consider two kinds of elements, vector space elements, and projective space elements.

If \(v\) is a vector in \(\omega\) fixed by \(H\), the map from \(L\) sending 1 to \(v\) is a map of \(H\) representations, hence is 0-definable. Since over \(\omega\), everything is in definable bijection with the field, and since \(G_m\) has no non-trivial torsors, every \(\omega\)-definable line has a \(\omega\)-definable point. Therefore, a line fixed by \(H\) comes from a fixed subspace of some representation, hence the inclusion map is also 0-definable. □

3.11. We note that conversely, given \(C\) and \(\omega\), the automorphism group \(G\) is a Kolchin closed subgroup of \(GL(V_X)\), where \(X\) generates \(C\) as a differential tensor category. Furthermore, by [Hru06], \(\omega\) gives an equivalence of categories between \(C\) and \(\text{Rep}_G\).

We can thus summarise the above results as follows.

3.12. **Theorem.** Let \(C\) be a differential rigid tensor category, \(\omega\) a fibre functor. Then \(\omega\) is a differential tensor equivalence between \(C\) and \(\text{Rep}_G\), where \(G\) is the pro-linear differential algebraic group \(\text{Aut}^\partial(\omega)\).

---

**References**


*Department of Maths, University of East-Anglia, Norwich, NR4 7TJ, England*

E-mail address: mailto:m.kamensky@uea.ac.uk

URL: [http://mkamensky.notlong.com](http://mkamensky.notlong.com)