PICARD–VESSIOT STRUCTURES

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Abstract. We demonstrate existence and uniqueness of Picard–Vessiot extensions satisfying prescribed properties, for systems of linear differential equations over a field satisfying the same properties, under some closure assumptions on the field of constants. An example includes the case of equations over a real field, with a real-closed field of constants. The result is obtained through a model theoretic approach.

Let $K$ be a differential field, with field of constants $C_K$. Let $Q$ be a linear differential equation of order $n$ over $K$. We recall that a Picard–Vessiot extension for $Q$ is a differential field extension $L$ of $K$ generated by $n$ independent solutions to $Q$, whose field of constants is $C_K$. We consider the following questions.

Question 0.1. Does $Q$ have a Picard–Vessiot extension (over $C_K$)?

Let $S$ be the universal first order theory of $K$ in the language of domains over $C_K$. For example, $S$ could be the theory of real fields.

Question 0.2. Is there a Picard–Vessiot extension whose universal theory is $S$?

Question 0.3. Are any two such Picard–Vessiot extensions isomorphic?

For instance, when $C_K$ is algebraically closed, the first and second questions coincide, and the answer to all questions is ‘yes’. In this paper, we prove the following.

Theorem 0.4. Assume that $C_K$ is existentially closed in $K$ (as a field). Then $Q$ admits a Picard–Vessiot extension.

This result occurs in [3, Thm 2.2]. We recall some definitions (see, e.g., [9]):

1. $C_K$ is existentially closed in $K$ if any constructible set over $C_K$ that has a point over $K$, also has a point in $C_K$.
2. If $C_K$ is existentially closed in any model of $S$ containing it, we say that $C_K$ is existentially closed.
3. If the collection of all existentially closed models of $S$ is axiomatisable by a first order theory, we call this theory the model companion of $S$.

Theorem 0.5. Assume that $S$ has a model companion $\hat{S}$, that $K$ is a model of $K$, and that $C_K$ is a model of $\hat{S}$.

Then there is a Picard–Vessiot extension which is a model of $S$. Furthermore, if $L_1$ and $L_2$ are two such extensions that induce the same $\hat{S}$-type on $K$, then $L_1$ is isomorphic to $L_2$. 
The last condition means the following. Since the $L_i$ are models of $S$, each can be embedded in a model $M$ of $\tilde{S}$. We require that such embeddings $t_i : L_i \rightarrow M$ can be found for which $\phi(t_1(u))$ holds in $M$ if and only if $\phi(t_2(u))$ holds there, for all $u \in K$ and all formulas $\phi$ of $\tilde{S}$. This is equivalent to asking that there are such embeddings that agree on $K$.

Setting $S$ to be the theory of real fields, or $p$-adic fields, we recover the main results of [3], [2] and [1] (in these cases, the model companions are the theories of real-closed or $p$-adically closed fields, respectively).

The proof is a variant of the approach of Bertrand and Deligne ([4], [9]) to the construction of the classical Picard–Vessiot extension. To outline it, it is convenient to slightly modify the terminology for Picard–Vessiot extensions: We remove the condition that the field of constants is $C_K$. The proof is then a combination of two main ideas.

The first idea is that the collection of Picard–Vessiot extensions is, essentially, classified by a constructible set $O$, in the sense that the $F$-points of $O$, for some field $F$ containing $C_K$, correspond to Picard–Vessiot extensions whose field of constants is contained in $F$. Thus, to find a Picard–Vessiot extension over $C_K$, it is enough to find a $C_K$-point of $O$. The uniqueness is obtained via a similar argument.

The second idea, is that, although it might not be easy to find such a point directly, it is easy to find a $K$-point of $O$ (this is achieved in [4] via the fibre functor that forgets the connection). The existential closure now implies that $O$ contains a $C_K$ point as well.

In our result we require the extension to satisfy additional properties, namely the axioms of a universal theory $S$. The collection of Picard–Vessiot extensions satisfying these properties can be viewed as a subset of $O$. This subset is no longer constructible, but it can be viewed as a definable subset in $\tilde{S}$. Once this is done, the same arguments go through.

In our model theoretic language, the second idea above is explained in [10], while the first one corresponds to [8]. The ideas of [8] were discovered by Hrushovski in [6], and hold in a completely abstract situation called internality. The connection between internality and the Tannakian formalism was discussed in [8].

To illustrate the argument, it is useful to consider the following simple example.

**Example 0.6.** We consider the equation $x' = \frac{x^2}{27}$ over the field $K = C(t)$, with $t' = 1$ (and $C$ is some field of constants). If $a$ is a non-zero solution (in some extension), it follows directly from the equation that $\frac{a^2}{27}$ is a non-zero constant $c$. The subfield generated by $a$ will thus be a Picard–Vessiot whose field of constants is the subfield generated by $c$ (over $C$). The set $O$ above, classifying the Picard–Vessiot extensions, can thus be taken to be the set of possible values for $c$, namely, the affine line with $0$ removed.

In this case, it is easy to find a point of $O$ in any field, but in general, we would not know in advance that $O$ has a $C$-point. However, it is easy to find a $K$-point: after all, $O$ is, by definition, the set of values of the function $y \mapsto \frac{y^2}{27}$, a function defined over $K$. We may simply evaluate this function on any non-zero element of $K$ (for instance 1). This shows that $O$ has a $K$-point. If $C$ is existentially closed in $K$, it will automatically have a $C$-point.

Assume now that $C$ (and therefore $K$) is a real field. We may then embed $K$ in some real closed field $M$. To do that, we need to choose an order on $K$, so suppose it was chosen so that $t > 0$. Which of the elements of $O$ correspond to
Picard–Vessiot extensions which are themselves real fields? In the extension, \( ct \) will have a square root, so, since \( t > 0 \), \( c \) should be positive as well. Hence we obtain the set of positive elements in \( \mathbf{O} \), a definable set in the theory of real closed fields.

What about uniqueness? For the uniqueness, we need to recall that \( \mathbf{O} \) is, in fact, the set of objects of a definable groupoid \( \mathbf{G} \). The morphisms of \( \mathbf{G} \) correspond to isomorphisms of Picard–Vessiot extensions, so two Picard–Vessiot over \( C \) are isomorphic precisely if there is a morphism over \( C \) in \( \mathbf{G} \) between the two corresponding objects. In this example, for \( p, q \in \mathbf{O} \), the set of morphisms from \( p \) to \( q \) is given by \( z^2 = \frac{p}{q} \). Thus, in \( RCF \), \( \mathbf{G} \) has two connected components. These two components correspond to the two choices of sign for \( t \). Furthermore, if \( C \) itself is a real closed field, any two Picard–Vessiot extensions over \( C \) of the same parity are isomorphic already over \( C \).

We remark that the method of this paper extends directly to other situations: several derivations, difference equations, etc. For concreteness, we stick to differential equations in one derivation.

1. Model theoretic setup

1.1. We view \( K \) as a structure for \( DCF \), the theory of differentially closed fields. We recall that this theory admits elimination of quantifiers. The given equation \( Q \) is viewed as a definable set in the theory \( DCF_K \), and we denote by \( C \) the definable set of constants \( x' = 0 \). Hence \( C_K = C(0) \), the 0-definable points of \( C \).

1.2. We let \( \bar{T} \) be the reduct of \( DCF_K \) to the definable sets \( Q \) and \( C \), with the full induced structure. We further let \( T \) be the reduct of that to \( C \).

**Lemma 1.3.** Any definable set in \( \bar{T} \) is the restriction (from \( DCF_K \)) of boolean combinations of polynomials over \( K \). In particular, the theory \( \bar{T} \) is precisely \( ACF_{C_K} \), and \( Q \) is a definable vector space over \( C \), of a fixed finite dimension \( n \) (possibly with more structure).

The theory \( \bar{T} \) admits quantifier elimination, hence is the model completion of \( \bar{T}_v \).

**Proof.** By elimination of quantifiers in \( DCF_K \), any definable set is equivalent to a boolean combination of differential polynomials. On both \( Q \) and \( C \), the derivative of an element is definable over that element by a (linear) polynomial, so can be substituted to obtain a usual polynomial. This also proves the quantifier elimination. \( \Box \)

1.4. By definition (and quantifier elimination in \( DCF \)), the boolean algebra of definable subsets of \( P = \mathbb{Q}^m \times \mathbb{C}^k \) in \( \bar{T} \) is generated by relations of the form \( R = R_X \), where \( X \) is a Zariski closed subset of \( \mathbb{P} = (\mathbb{A}^n)^m \times \mathbb{A}^k \). We call such a relation Zariski closed. The closed set \( X \) restricting to \( R \) need not be unique, but the collection of all Zariski closed \( X \) restricting to \( R \) has, by Noetherianity, a minimal element \( \mathbb{L}(R) \), which we call the minimal lift of \( R \). Thus, we have a bijection between Zariski closed subsets of \( P \) and minimal lifts in \( P \).

**Lemma 1.5.** The assignment \( R \mapsto \mathbb{L}(R) \) extends to a boolean algebra homomorphism from definable subsets of \( \mathbb{Q}^m \times \mathbb{C}^k \) to Zariski constructible subsets of \( (\mathbb{A}^n)^m \times \mathbb{A}^k \) (which is a section of the restriction).
From standard facts on constructible sets, the statement reduces to proving:

\[ K \subseteq L \]

More geometrically, the last proof says the following. Let \( ^\exists \)

This is true abstractly: Lemma \ref{corollary}. Let \( X \subseteq K \) and \( Y \subseteq L \) be subvarieties of \( ^\exists \) algebraic, so \( X \subseteq K \) can satisfy the equation \( x' = s(x) \) only if \( (x, s(x)) \in \tau X \). The subset of \( X \) given by the last condition is algebraic, so \( s \) will restrict to a vector field on the minimal lift of \( X \cap P \). Given two such subvarieties \( X_i \), we have that the same vector field restricts to both, hence we have a vector field on their intersection (at least, if they intersect transversally). The axioms of \( DCF \) now imply that the intersection admits a trajectory.

Remark 1.6. More geometrically, the last proof says the following. Let \( \mathcal{P} \) be a variety over \( K \), and let \( s : \mathcal{P} \to \tau \mathcal{P} \) be a (twisted) vector field over \( K \). Let \( \mathcal{P} \) be the definable set given within \( \mathcal{P} \) by \( x' = s(x) \). The points of \( \mathcal{P} \) in some differential field can be viewed as (formal) trajectories on \( \mathcal{P} \) to \( s \).

Now, if \( X \) is a subvariety of \( \mathcal{P} \), a point \( x \in X \) can satisfy the equation \( x' = s(x) \) only if \( (x, s(x)) \in \tau X \). The subset of \( X \) given by the last condition is algebraic, so \( s \) will restrict to a vector field on the minimal lift of \( X \cap \mathcal{P} \). Given two such subvarieties \( X_i \), we have that the same vector field restricts to both, hence we have a vector field on their intersection (at least, if they intersect transversally). The axioms of \( DCF \) now imply that the intersection admits a trajectory.

1.7. Lemma \ref{corollary} may be reformulated as saying that \( X \mapsto \mathbb{L}(X) \) is an interpretation of the universal part of \( \mathcal{T} \) in \( \mathcal{T}_K \). In particular, if \( F \) is a \( \mathcal{T}_K \)-structure (i.e., a field extension of \( K \)), we may consider the pullback structure \( \mathcal{F} \) for \( \mathcal{T} \), given by \( X(\mathcal{F}) = \mathbb{L}(X)(F) \). We now extend the statement to the full theory.

Corollary 1.8. The assignment \( X \mapsto \mathbb{L}(X) \) from \ref{corollary} determines an interpretation of \( \mathcal{T} \) in \( \mathcal{T}_K \), whose restriction to \( \mathcal{T} \) is the extension of scalars.

Proof. This is true abstractly: Lemma \ref{corollary} asserts this on the level of the universal parts, and the full statement follows since both theories are model-complete.

In more details, since \( \mathcal{T} \) is model-complete, it suffices to prove that if \( F \) is a model of \( \mathcal{T}_K \) (i.e., an algebraically closed field extending \( K \)), then the pullback structure \( \mathcal{F} \) is existentially closed (cf. \cite[Prop. 2.39]{kamensky}). But any \( \mathcal{T} \)-structure \( M \) extending \( \mathcal{F} \) can be viewed, via \( \mathbb{L} \), as a structure for a reduct of \( \mathcal{T}_K \), extending \( F \). Hence, if \( X \) is a quantifier-free set over \( \mathcal{F} \) in \( \mathcal{T} \) with a point in \( M \), it will also be a point of \( \mathbb{L}(X) \). Since \( F \) is existentially closed, \( \mathbb{L}(X) \) will have a point in \( F \), which will be a point of \( X \) in \( \mathcal{F} \).

In the next section, we will recall the abstract relationship between \( \mathcal{T} \) and \( \mathcal{T} \). In particular, using that terminology, the following lemma says that \( \mathcal{T} \) admits a Picard–Vessiot structure over \( K \), hence that \( \mathcal{T} \) is internal to \( \mathcal{T} \).

Lemma 1.9. Over \( K \), any definable set in \( \mathcal{T} \) is isomorphic to a subset of \( \mathcal{C} \), by an isomorphism fixing \( C \) pointwise.

Proof. We need only to define a bijection from \( \mathcal{C}^n \) to \( \mathbb{Q} \), but this is given by the standard basis of \( K^n \) over \( K \).
Remark 1.10. Instead of working with $\hat{T}$ as we defined it, we could take a bigger reduct, containing also a sort for the image of each polynomial function on products of $\mathbb{Q}$ and $\mathbb{C}$, and function symbols for such functions. Such a construction would essentially give an explicit description of $\hat{T}^eq$. This approach is closer to the approach through the Tannakian formalism.

2. Internality

In this section we discuss some aspects of internality, the abstract model theoretic notion controlling the situation described in [G]. See [G] for details.

2.1. Let $\mathcal{T}$ be a complete theory with elimination of imaginaries. We consider expansions $\tilde{T}$ of $\mathcal{T}$, and when doing so, identify $\mathcal{T}$ with a definable set (or a collection of them) in $\tilde{T}$ (the situation is only interesting when $\tilde{T}$ has additional sorts). We will say that a definable set in $\tilde{T}$ comes from $\mathcal{T}$ if it is obtained from $\mathcal{T}$ via this identification.

We assume that $\mathcal{T}$ is stably embedded in $\tilde{T}$: any parametrically definable subset of a definable set coming from $\mathcal{T}$ is definable, using parameters, in $\mathcal{T}$.

If $K$ is a $\tilde{T}$-structure (i.e., a definably closed set), we denote by $K_0$ its restriction to $\mathcal{T}$. We say that $K$ is over a $\mathcal{T}$-structure $L$ if $K_0 \subseteq L$.

2.2. Types over $\mathcal{T}$. Let $X$ be a definable set in $\tilde{T}$, and let $Z \subseteq X \times C$ be a definable subset, where $C$ comes from $\mathcal{T}$. If $x$ is an element of $X$ (in some model), the $x$-definable subset

$$ Z_x := \{ c \mid (x, c) \in Z \} \subset C $$

is defined, by stable-embeddedness of $\mathcal{T}$, over a parameter $d$ in $\mathcal{T}$, which can be taken canonical by elimination of imaginaries. We call this parameter $d$ the $Z$-type of $x$ over $\mathcal{T}$, and denote it $tp_Z(x/\mathcal{T})$.

The function $x \mapsto tp_Z(x/\mathcal{T})$ is definable, and we denote its image by $St_Z(X/\mathcal{T})$ (the $Z$-Stone space of $X$ over $\mathcal{T}$). Hence, it is a definable set in $\mathcal{T}$, whose $L$-points (for $L$ a $\mathcal{T}$-structure) correspond to those $Z$-types over $L$ (in the usual sense) that have a unique extension to any $\mathcal{T}$-structure containing $L$.

The construction of the stone-space above is clearly functorial in $Z$ with respect to definable maps over $X$. In particular, given $Z_i \subseteq X \times C_i$ (for $i = 1, 2$), one may form their fibre product $Z$ over $X$, and obtain definable maps $St_Z \to St_{Z_i}$. The collection of all $Z \subseteq X \times C$, for all $C$ coming from $\mathcal{T}$, determines in this way a filtering inverse system $St(X/\mathcal{T}) = (St_Z(X/\mathcal{T}))/Z$, which we call the Stone space of $X$ over $\mathcal{T}$. Hence, it is a pro-definable set in $\mathcal{T}$, classifying types in $X$ over $\mathcal{T}$, as before.

We note that if $L$ is a $\mathcal{T}$-structure, and $x$ is an element of $X$ (both inside some model of $\tilde{T}$), then the type of $x$ over $\tilde{T}$ is in $St(X/\mathcal{T})(L)$ precisely if the restriction of $dcl_{\tilde{T}}(x)$ to $\mathcal{T}$ is contained in $L$.

2.3. Internal covers. We recall from [G] (and the discussion in [G]) that an internal cover of $\mathcal{T}$ is an expansion $\tilde{T}$ as above, such that for some $\tilde{T}$-structure $K$, the induced interpretation $\tilde{T}_{K_0} \to \tilde{T}_K$ is a bi-interpretation. A structure $K$ with this property will be called a Picard–Vessiot $\mathcal{T}$-structure (over $L$, if $K_0 \subseteq L$ for a given $\mathcal{T}$-structure $L$).
2.4. **Definable groupoids.** As explained in [G], to any internal cover \( \tilde{T} \) of \( T \), it is possible to associate a (pro-)definable groupoid \( G \) in \( T \). The set of objects of \( G \) is the stone space \( O = \text{St}(X/T) \), for a particular set \( X \) in \( \tilde{T} \) (on which \( G \) depends). For each definable set \( Q \) of \( T \), \( X \) parametrises a family of injective maps from \( Q \) to a set coming from \( T \). For any element \( x \) of \( X \), the image \( Q_x \) of the injective map determined by \( x \) is defined over \( x \), hence (being a definable set in \( T \)) over the type \( p \in O \) of \( x \) over \( T \). In particular, it is the same for all elements satisfying \( p \), and will be denoted \( Q_p \).

The definable set of morphisms from \( p \) to another such type \( q \) is, by definition, the set of equivalence classes of pairs \((x, y)\), realising \( p \) and \( q \), where two pairs are equivalent if the composed bijection \( Q_x \xrightarrow{y_{x^{-1}}} Q_y \) is the same (for all definable sets \( Q \) of \( T \)). Hence, each morphism \( u \) from \( p \) to \( q \) determines a \( u \)-definable bijection \( Q_p \to Q_q \), and composition is defined by composing these bijections.

2.5. **Picard–Vessiot interpretations.** Given any \( T \)-structure \( L \), we let \( I_L = \text{Hom}_T(\tilde{T}, T_L) \) be the category with objects interpretations of \( \tilde{T} \) in \( T_L \), whose restriction to \( T \) is the expansion by \( L \). The morphisms are given by compatible \( T_L \)-definable isomorphisms, whose restriction to \( T \) is the identity.

We note that any such interpretation determines a \( \tilde{T} \)-structure \( \tilde{L} \), whose restriction to \( T \) is \( L \). A **Picard–Vessiot interpretation** over \( L \) is an interpretation as above where \( \tilde{L} \) is Picard–Vessiot, in the sense of [G]. Conversely, if \( K \) is a Picard–Vessiot structure, it determines, by definition, an interpretation of \( \tilde{T} \) in \( T_K \), such that \( K \) is isomorphic (over \( K_0 \)) to \( \tilde{K}_0 \). Hence we have an equivalence between Picard–Vessiot structures, and Picard–Vessiot interpretations, over the same base.

**Proposition 2.6.** Let \( G \) be a definable groupoid associated to the internal cover \( \tilde{T} \) of \( T \). Then for any \( T \)-structure \( L \), there is a faithful embedding \( r_L : G(L) \to \text{Hom}_T(\tilde{T}, T_L) \). Furthermore:

1. These embeddings commute, on the level of morphisms, with the action of automorphism groups of (models of) \( T \).
2. Each \( r_L \) is full.
3. If \( p \) is an object of \( G \) over \( L \), then the corresponding interpretation \( r_L(p) \) is Picard–Vessiot. Conversely, for any finite set \( S \) of Picard–Vessiot interpretations over \( L \), \( G \) can be chosen so that \( S \) is in the essential image of \( r_L \).

In particular, if \( L \) is a model of \( T \), then \( r_L \) is an equivalence.

**Proof.** The functor is given by the association \( p \mapsto (Q \mapsto Q_p) \) described in [G]. It is clear from the above description that this is a faithful functor.

1. This is obvious, since the application of a morphism of \( G \) is given by a first order formula over \( L \).
2. Let \( f \) be an isomorphism over \( L \) between \( r_L(p) \) and \( r_L(q) \). By [H], it suffices to show that \( f \) comes from a morphism in \( G(M) \), where \( M \) is a (somewhat) saturated model containing \( L \). Since \( G \) is connected, there is, in \( G(M) \), a morphism \( x \) from \( q \) to \( p \). Composing \( f \) with \( r_L(x) \), we reduce to the case \( q = p \), which is classical.
3. If \( p \) is an object of \( G(L) \), then \( r_L(p) \) identifies the definable set \( X \) (considered in [G]) with the definable set \( \text{Mor}_G(p, -) \), so that the realisations of a type \( q \) are identified with \( \text{Mor}_G(p, q) \). In particular, the realisations of \( p \) are...
identified with $\text{Aut}_G(p)$. Hence, $p$ is realised in $\hat{L}$ by the identity element $1 \in \text{Aut}_G(p)$. This element determines, in $\mathcal{T}_L$, an isomorphism between $r_L(p)$ composed with the expansion from $\mathcal{T}_L$ to $\mathcal{T}_L$ and the identity on $\mathcal{T}_L$.

Conversely, assume first that $S$ consists of one Picard–Vessiot interpretation. Then we may choose $X$ above to contain a $L$-point. For more than one point, since all groupoids corresponding to a given cover are equivalent, we may find a groupoid containing the groupoids corresponding to each element of $S$.

The last statement follows since $\hat{L}$ is also a model in this case. \hfill \qed

Proposition 3.8 is all we need for the application, which continues in the following section. In the current section, we describe how to obtain the full category $\text{Hom}_T(\mathcal{T}, \mathcal{T}_L)$ canonically from $G$.

2.7. Let $\mathcal{T}$ be an internal cover, $G$ a definable groupoid in $T$ corresponding to it, and let $M$ be a saturated model of $\mathcal{T}$. For $L$ a $\mathcal{T}$-structure, let $H = \text{Aut}(M/L)$, and for an element $x \in M$, let $H_x$ be the stabiliser of $x$.

We consider the category $G(L)$ whose objects are pairs $(x, \Phi)$, where $x$ is an object of $G$ over $M$, and $\Phi : H/H_x \rightarrow \text{Mor}_G(M)$ is a function, whose pullback to $H$ we denote by $\tau \mapsto g_\tau$. We require that all $g_\tau$ have codomain $x$, and that for all $\tau, \sigma \in H$,

$$g_{\tau \sigma} = g_\tau \circ \tau(g_\sigma) \quad (1)$$

This includes the requirement that the morphisms on the right are composable. Hence, for any $\tau$, the domain of $g_\tau$ is $\tau(x)$, and $g_\tau$ is the unique element in the image of $\Phi$ with this domain. Also, $g_1$ is the identity on $x$. The morphisms are systems of morphisms in $G(M)$ that make the obvious diagrams commute.

There is a fully faithful functor $G(L) \rightarrow G(L)$, assigning to each object $p$ of $G(L)$ the constant function $\tau \mapsto 1_p$. We claim that the functor $r_L$ extends to an equivalence of categories from $G(L)$ to $I_L := \text{Hom}_T(\mathcal{T}, \mathcal{T}_L)$ (in other words, $L \mapsto I_L$ is the stack associated to the prestack $L \mapsto G(L)$ determined by $G$).

To show this, first note that this is true for $L = M$ by the proposition. Thus, it is enough to prove the same statement, but with $G$ replaced by $L \mapsto I_L$. Given an object $\tau \mapsto d_\tau$ of $I_L$, where $d_\tau : i_\tau \rightarrow i_0$ is an isomorphism of interpretations over $M$, we get an actual interpretation

$$\lim_{\{d_\tau | \tau \in H/H_0\}} i_\tau \quad (2)$$

defined over $L$, and isomorphic to the given system over $M$.

Example 2.8. Let $G$ be a group definable in $\mathcal{T}$, and let $\mathcal{F}$ be the theory of $G$-torsors. Thus, $\mathcal{F}$ is obtained from $\mathcal{T}$ by adding a new sort $X$, and function symbol for an action of $G$ on $X$, which $\mathcal{F}$ implies to be free and transitive. Then $\mathcal{F}$ is an internal cover of $\mathcal{T}$, and $G$ is the corresponding groupoid (with one object). This is a special case of the construction in 3.8.

If $L$ is a $\mathcal{T}$-structure, then $I_L$ is the category of $L$-definable $G$-torsors in $\mathcal{T}$. In particular, it has, in general, more than one object. The functor $r_L$ assigns the trivial torsor to the only object of $G$. If $L$ is a model, $r_L$ is an equivalence, since all torsors then have points, and so become isomorphic. It is well known, in this case, that $L \mapsto I_L$ is indeed the stack associated to $G$. The general case is similar.
To illustrate the non-canonicity of the groupoid, assume now that over the definable group $G_0$ in $T$, there is a non-trivial torsor $P$. Let $G_1$ be the group of definable automorphisms of $P$, as a $G_0$-torsor (so $G_1$ is isomorphic to $G_0$ over any point of $P$, but not over 0). Consider the groupoid $G$ with two (named) objects 0 and 1, with $\text{Mor}_G(i, i) = G_i$, and $\text{Mor}_G(0, 1) = P$ (with composition given by the torsor structures). A cover $\tilde{T}$ corresponding to it has two additional sorts, $U_0$ and $U_1$, with $U_i$ a $G_i$-torsor, and a map $m : U_0 \times U_1 \to P$ (satisfying suitable properties). There are two interpretations of $\tilde{T}$ in $T$, the first interprets $U_0$ as $G_0$, $U_1$ as $P$, and $m$ as the action, and the other assigns $P$, $G_1$ and the other action to $U_0$, $U_1$ and $m$. These two interpretations correspond to the two objects of $G$. However, each of the groups separately also represents the same cover, and choosing just the group as $G$, we miss one of the objects.

**Remark 2.9.** More generally, it is possible to replace $T_L$ in the above picture by an arbitrary interpretation $i : T \to T_1$. The statement then provides a fully faithful functor from the 0-definable points of $i(G)$ in $T_1$, to the interpretations of $\tilde{T}$ in $T_1$, over $T$.

### 3. Applying internality

We now return to the situation of $\mathfrak{G}$. Thus, to a linear differential equation over a differential field $K$ we associate a theory $T$, expanding the theory $T = ACFC_{C_K}$. As mentioned in §2, Lemma 2.3 implies that $\tilde{T}$ is an internal cover of $T$, which admits a Picard–Vessiot structure over $K$ (viewed as a pure field). We will use the geometric description of such structures in §2 to descend it to a Picard–Vessiot structure over $C_K$.

**3.1. Proof of Theorem 1.3.** Let $i$ be the Picard–Vessiot interpretation of $\tilde{T}$ in $T_K$ provided by Lemma 1.4. According to Proposition 2.4, we may find a definable groupoid $G$ in $T$, corresponding to the internal cover $\tilde{T}$, and a $K$-point of its definable set $O$ of objects, corresponding to $i$. Since $C_K$ is existentially closed in $K$, it follows that $O$ has a $C_K$-point. Again by Proposition 2.4, this point corresponds to a Picard–Vessiot interpretation over $C_K$, which corresponds to a Picard–Vessiot extension over $C_K$, by 2.5.

**3.2.** To go further, let $S$ be the universal theory of $K$ in the language of domains over $C_K$. To answer the other main questions, we will obtain a variant of Proposition 2.4, obtaining a definable groupoid that classifies Picard–Vessiot interpretations such that the corresponding structure is a model of $S$. Most of the work can be done abstractly, in the setting of §2, but for concreteness, we proceed with differential fields.

We fix a definable groupoid $G$ in $T$, corresponding to the cover $\tilde{T}$ of $T$. We denote its definable set of objects by $O$. For any $T$-structure $L$, and any $a \in O(L)$, we denote by $\omega_a$ be the interpretation of $T$ in $T_L$ corresponding to $a$.

**Lemma 3.3.** For any quantifier-free formula $\phi(x)$ of $\tilde{T}$, there is a universal formula $O_\phi$ of $T$ (over 0), defining a subset of $O$, with $O_\phi(L)$ the set of $a \in O(L)$ for which $\forall x \phi(x)$ holds in $\omega_a(L)$.

We remark that the universal quantifier in $O_\phi$ is interpreted with respect to $L$. 
Proof. The formula $\phi(x)$ corresponds to a family $\psi(y, w)$, where $w$ ranges over $O$, and $\psi(y, a)$ is the interpretation of $\phi$ according to $\omega_n$. Hence $O_\phi$ is given by $\forall y \psi(y, w)$. □

3.4. Let $S$ be a universal theory extending the universal part of $T$ (in the language of fields). We obtain a universal theory $\tilde{S}$, extending the universal part of $\tilde{T}$, given by

$$\tilde{S} = \{ \forall x \phi(x) | \forall x \mathbb{L}(\phi)(x) \in S \}$$

(3)

where $\mathbb{L}$ is the interpretation given by Lemma 3.2. Applying Lemma 3.2 to each element of $S$, we conclude that the set $O_\mathbb{S}(L)$ of $a \in O(L)$ with $\omega^*_{\mathbb{S}}(L)$ a model of $\tilde{S}$ is (infinitely) definable, by a collection of universal formulas.

Example 3.5. Consider again Example 0.6. In this case, $O$ is the affine line with 0 removed, with $\omega_n$ the interpretation in which the value of the function $\frac{x^2}{t}$ (from $\mathbb{Q}$ to $\mathbb{C}$) on $1 \in \omega_n(Q)$ is $a$.

Let $\phi(q, c)$ be the formula

$$\frac{q^2}{t} + c^2 = 0 \implies q = c = 0$$

(4)

of $\tilde{T}$. The family $\psi$ occurring in the proof of Lemma 3.2 is then given by

$$ad^2 + c^2 = 0 \implies d = c = 0$$

(5)

so $O_\phi$ is the set of all $a$ with $-a$ not a square.

3.6. Proof of Theorem 0.5. We fix a universal domain $M$ for $\tilde{S}$, containing $K$. By §3.2, there is a definable subset $O_S$ of the set $O$ of objects of the groupoid corresponding to the equation, whose points are the Picard–Vessiot extensions satisfying $S$.

We claim that $O_S$ is non-empty. As in §3.1, the tautological interpretation given by Corollary 1.8 produces a $K$-point of $O$. We recall that such an object is obtained by applying certain definable functions to any basis of the vector space determined by the differential equation (these are the functions that assign to each such basis its type over $C$, as explained in §2.2). But these definable functions are definable in the pure (quantifier free) field structure, so applying them to any basis over $K$ gives a $K$-point of $O_S$.

Now the proof continues as in §3.1. The set $O_S$ is defined over $C(K)$, and has a point in $K$ (a model of $S$). Since $C(K)$ is existentially closed, this set has a point in $C(K)$, corresponding to a Picard–Vessiot extension over $C(K)$, satisfying $S$.

Uniqueness is similar: Given an embedding of $K$ in $M$ (corresponding to an extension of the field structure on $K$ to a full type), any two Picard–Vessiot extensions over $C(K)$ satisfying $S$ correspond to points $a, b \in O_S(C(K))$. The general theory of internal covers says that the groupoid is connected, so $\text{Mor}(a, b)$ is a non-empty definable set over $C(K)$ (in $\tilde{S}$). Hence it has a point in $C(K)$. □

Remark 3.7. The same proof shows that the statement holds more generally when $S$ is a Robinson theory (and $C(K)$ is a universal domain).

References


