Introduction to Singularities, 201.1.0361

Homework 1

Spring 2017 (D.Kerner)

(1) For the given functions $(\mathbb{R}^n, 0) \xrightarrow{f} \mathbb{R}$ rectify the set $f^{-1}(0)$ and the graph of f. (Namely, show an explicit local change of variables that sends $f^{-1}(0)$ and the graph to linear spaces)
i. $f(x, y, z) = z - e^y + cos(x)$, ii. $f(x, y) = x^2 + x - y^2$, iii. $f(x, y) = sin(x) + ln(1 + x^2 + y^2)$.

(2) Draw the sets $f^{-1}(0) \subset \mathbb{R}^2$ in the following cases:
i. $f(x, y) = xy$, ii. $f(x, y) = x^2 - y^2$, iii. $f(x, y) = y^2 - x^{2n}$, iv. $f(x, y) = y^2 - x^{2n+1}$,
v. $f(x, y) = y^2 - x^2 - x^3$, (here google: "nodal cubic")

(3) Draw the sets $f^{-1}(0) \subset \mathbb{R}^3$ in the following cases:
i. $f(x, y, z) = xy$, ii. $f(x, y, z) = xyz$, iii. $f(x, y, z) = (z - x^2)z$, iv. $f(x, y, z) = (z - x^2 - y^2)z$,
v. $f(x, y, z) = z^2 - x^3$, vi. $f(x, y, z) = y^2 - x^2 z$, (here google: "Whitney umbrella")

(4) For many pictures of singular curves/surfaces go to: *https://imaginary.org/galleries*

(5) Let R be one of the following:
i. $C^\infty(\mathbb{R}^n, 0) = \{$infinitely differentiable functions defined on some open neighborhoods of $0 \in \mathbb{R}^n\}$. (Note: each function is defined on its own open neighborhood.)
ii. $\mathbb{R}[[x_1, \dots, x_n]] = \{$formal power series in variables $x_1, \dots, x_n$, with real coefficients$\}$. And similarly $\mathbb{C}[[\underline{x}]]$.
iii. $\mathbb{R}\{\underline{x}\} = \{$power series in variables $x_1, \dots, x_n$ that converge in some open neighborhoods of $0 \in \mathbb{R}^n\} \subset \mathbb{R}[[\underline{x}]]$. (The convergence as was defined in the lecture. Note: each power series has its own domain of convergence.) And similarly $\mathbb{C}\{\underline{x}\}$.

(a) Prove: R is a commutative, associative ring with a unit. (Let $\mathcal{D}_f, \mathcal{D}_g$ be the domains of definition for f, g. What are the domains of definition for $f + g, f \cdot g$?)
(b) Define $\mathfrak{m} := \{f \in R | \ f(0) = 0\}$. Prove that $\mathfrak{m} \subseteq R$ is an ideal.
(c) For $f \in R$ prove: if $f(0) \neq 0$ then $\frac{1}{f} \in R$. (For the cases $R = \mathbb{R}\{\underline{x}\}$, $\mathbb{C}\{\underline{x}\}$ one needs some auxiliary results from analysis, you can omit these cases meanwhile.)
(d) Prove that $\mathfrak{m}$ is a maximal ideal in R, i.e. if an ideal $I \subseteq R$ satisfies $I \supsetneq \mathfrak{m}$, then $I = R$. Can you demonstrate some nice/simple generators of $\mathfrak{m}$?
(e) Prove that $\mathfrak{m}$ is the unique maximal ideal in R. (A ring that has a unique maximal ideal is called "local", geometrically it corresponds to a "small" open neighborhood of a point.)
(f) The Jacobian ideal of an element $f \in R$ is defined by $Jac(f) = \langle \partial_1 f, \dots, \partial_n f \rangle \subset R$. Prove: $Jac(f) \neq R$ (and thus $Jac(f) \subseteq \mathfrak{m}$) iff 0 is a critical point of f.

(6) Draw $f^{-1}(\epsilon)$ for $\epsilon > 0$, $\epsilon = 0$, $\epsilon < 0$ in the following cases:
i. $f(x) = x^p$ (here $p = 1$ or $p \geq 2$; distinguish between p odd and even), ii. $f(x, y) = x^2 - y^2$,
iii. $f(x, y) = x^2 + y^2$, iv. $f(x, y, z) = x^2 + y^2 + z^2$, v. $f(x, y, z) = x^2 + y^2 - z^2$.