Introduction to Singularities, 201.1.0361

Homework 2

Spring 2017 (D.Kerner)

Let the field $\Bbbk$ be one of $\mathbb{R}$, $\mathbb{C}$, let the ring R be one of $C^\infty(\mathbb{R}^n,0)$, $\Bbbk[[\underline{x}]]$, $\Bbbk\{\underline{x}\}$.

(1) (a) Find $ord(f)$ in the following cases: i. $f = sin^p(sin^q(x))$, ii. $f = ln(1+x^p)$.

(b) Prove the equivalent definitions of $ord(f)$:

i. $ord(f) = p$ iff $f(0) = f^{(1)}|_0 = \cdots = f^{(p-1)}|_0 = 0$ and $f^{(p)}|_0 \neq 0$.

ii. $ord(f) = sup\{p|\ f \in (\underline{x})^p,\ f \notin (\underline{x})^{p+1}\}$.

(c) Prove the basic properties of $ord(f)$:

i. Suppose $R \neq C^\infty(\mathbb{R}^n,0)$. Then $ord(f) = \infty$ iff $f \equiv 0$. (For the ring $R = C^\infty(\mathbb{R}^n,0)$ there are many elements (functions flat at the origin) with $ord(f) = \infty$.)

ii. $ord(f \cdot g) = ord(f)ord(g)$, $ord\frac{f}{g} = ord(f) - ord(g)$.

iii. $ord(f \pm g) \geq min(ord(f), ord(g))$, give an example with strict inequality.

iv. If $f \overset{\mathcal{K}}{\sim} g$ then $ord(f) = ord(g)$.

(2) (a) Let $f \in C^\infty(\mathbb{R}^1,0)$ and suppose $ord(f) > p$. Prove that $\frac{f(x)}{x^p} \in C^\infty(\mathbb{R}^1,0)$. Prove that $\sqrt[p]{1 + \frac{f_{>p}(x)}{x^p}} \in C^\infty(\mathbb{R}^1,0)$.

(b) Suppose $f \in \mathfrak{m} \subset R$, prove that $\sqrt[p]{1+f} \in R$. (You can use the implicit function theorem for all our choices of R.)

(3) (a) Check that $\mathcal{R}$ is indeed an equivalence relation on R.

(b) Recall that a change of variables, $\phi \circlearrowleft (\mathbb{R}^n,0), (\mathbb{C}^n,0)$, induces an automorphism of the local rings, making the diagram commute. Obtain from here the criterion: the Milnor algebra is an invariant of $\mathcal{R}$-equivalence. (In particular, its dimension, as a vector space, is invariant.)

$$\begin{array}{ccc} R & \overset{\phi^*}{\to} & R \\ \cup & & \cup \\ Jac(f) & \to & Jac(\phi^*(f)) \end{array}$$

(c) Which of the following are $\mathcal{R}$-equivalent? (You can use the implicit function theorem in R.) In each case compute the Milnor number.

i. $y^2 - x^n$, ii. $y^2 + y^3 - x^n - x^{n+1}$, iii. $y^2 - yx^n - x^{2n}$, iv. $y^3 - x^n$, v. $y^3 - yx^n$.

(4) (a) Check that $\mathcal{K}$ is indeed an equivalence relation on R.

(b) Let $f, g \in R$, with $f(0) = 0$ and $f \overset{\mathcal{K}}{\sim} g$. Prove that 0 is a critical point of f iff it is a critical point of g. Is the condition $f(0) = 0$ necessary here?

(c) (Here $\Bbbk = \mathbb{C}$.) Let $f_t = x^p + y^q + z^r + txyz$, with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Prove that $f_t \overset{\mathcal{K}}{\sim} f_1$ for any $t \neq 0$. What is the condition on t, t' to ensure $f_t \overset{\mathcal{R}}{\sim} f_{t'}$?
(In question 3.c. you saw countably many distinct $\mathcal{R}$-types. Here you see a continuum of distinct $\mathcal{R}$-types. Later we will see examples with continua of distinct $\mathcal{K}$-types. C'est la vie.)

(5) (a) Suppose $f(0) = 0$ and 0 is a critical point of f. Consider the matrix of second derivatives, $f^{(2)}|_0$. Write down the transformation rule for this matrix under $\mathcal{R}$, $\mathcal{K}$-equivalences. Prove that $rank(f^{(2)}|_0)$ is $\mathcal{K}$-invariant.

(b) Prove that $f^{(2)}|_0$ can be diagonalized by a local change of coordinates. (In Linear Algebra you saw that a symmetric matrix can be diagonalized by $A \to U^t A U$. Here, if A is over $\mathbb{R}$, then so is U. In particular, all the eigenvalues of A are real.)

(c) Given a symmetric matrix, $A \in Mat^{sym}_{n\times n}(\mathbb{R})$, denote by (n_+, n_0, n_-) the number of its positive/zero/negative eigenvalues. Prove that the triple (n_+, n_0, n_-) is preserved under $\mathcal{R}$-equivalence. (Though the eigenvalues are not preserved.) Prove that the numbers $n_0, |n_+ - n_-|$ are preserved under $\mathcal{K}$-equivalence.

(d) Let $p(\underline{x})$ be a polynomial of degree 2, not necessarily homogeneous. Prove that there exists an affine change of coordinates, $\underline{x} \to U \cdot \underline{x} + \underline{v}$, (for some $U \in Mat_{n\times n}(\Bbbk)$ and some $\underline{v} \in \Bbbk^n$) that brings $p(\underline{x})$ to the form: $\sum\limits_{i=1}^{r} (\pm) x_i^2 + c_1 x_{r+1} + c_0$. Here:

(i) $(\pm)$ are only for the case $\Bbbk = \mathbb{R}$;

(ii) c_1 is either 0 or 1;

(iii) if $c_1 \neq 0$ then $c_0 = 0$.

(e) Use this to classify all the curves of degree 2 in $\mathbb{R}^2$. What about the surfaces of degree 2 in $\mathbb{R}^3$?