# Introduction to Singularities, 201.1.0361

## Homework 3

Spring 2017 (D.Kerner)

Let the field $\mathbb{k}$ be one of $\mathbb{R}, \mathbb{C}$, let the ring $R$ be one of $C^\infty(\mathbb{R}^n, 0)$, $\mathbb{k}[[\underline{x}]]$, $\mathbb{k}\{\underline{x}\}$.

(1) (over $\mathbb{C}$) Prove: $f$ is complex-differentiable at $pt$ iff $f(\underline{z}) = f(pt) + \sum\limits_{j=1}^{n} a_j(\underline{z})(z_j - pt_j)$, where $\{a_j(\underline{z})\}$ are continuous.

(2) In the lecture, in the proof of implicit function theorem, we showed: if the matrix $\{\partial_{x_j} F_i(\underline{x}, \underline{y})\} \in Mat_{m\times m}(\mathbb{C})$ is non-degenerate, then so is the $2m \times 2m$ matrix composed of the real and imaginary parts. The proof was by a direct computation. This is also implied by any of the following statements, prove them.

Given a linear map of $\mathbb{K}$-vector spaces, $V \stackrel{L}{\to} W$ and a subfield $\mathbb{k} \subset \mathbb{K}$. Consider $V, W$ as vector spaces over $\mathbb{k}$, then we have the induced $\mathbb{k}$-linear map $V_\mathbb{k} \stackrel{(L)_\mathbb{k}}{\to} W_\mathbb{k}$ .

- (a) $L$ is surjective iff $(L)_\mathbb{k}$ is surjective.
- (b) $L$ is injective iff $(L)_\mathbb{k}$ is injective.

(3) We proved the implicit function theorem for the ring $\mathbb{C}\{\underline{x}\}$. Obtain from here the implicit function theorem for the ring $\mathbb{R}\{\underline{x}\}$.

(4) (We proved the Morse lemma in the class. You can use the ideas in the proof for the questions below.)

- (a) Let $g_1 \in (x^2) \cap \mathfrak{m}^3$, $g_2 \in (y^2) \cap \mathfrak{m}^3$. Construct a change of coordinates that brings $x^2 + y^2 + g_1 + g_2$ to $x^2 + y^2$.
- (b) Construct a change of variables that brings $x^2 + y^2 + z^2 + xyz$ to $x^2 + y^2 + z^2$.
- (c) Prove: if $o$ is a non-degenerate critical point of some $f \in R$, and $f(o) = 0$, then $f \stackrel{\mathcal{R}}{\sim} \sum(\pm)x_i^2$.
- (d) More generally, for the matrix of second derivatives, suppose $rank(f^{(2)}|_o) = r$. Prove: $f \stackrel{\mathcal{R}}{\sim} \sum\limits_{i=1}^{r}(\pm)x_i^2 + f_{\geq 3}(x_{r+1}, \dots, x_n)$, where $f_{\geq 3}(x_{r+1}, \dots, x_n) \in \mathfrak{m}^3$.

(5) Compute the Milnor/Tjurina numbers for the following critical points:

i. $f(x, y) = y^p + x^q$, ii. $f(x, y) = y^p x + x^q$, iii. $f(x, y) = y^4 + y^2 x + x^4$.
iv. $f = \frac{x^5}{5} + \frac{y^5}{5} + \frac{x^3 y^3}{3}$.

(6)

- (a) In the class we observed: if $f \in \mathfrak{m}^3$ then $\mu(f) \geq n + 1$. Use the same methods to prove a stronger result: if $ord(f) = p$ and $dim(R) = n$ then $\mu(f) \geq \binom{n+p-1}{p-1} - n$.
- (b) Classify (up to $\mathcal{R}$-equivalence) the critical points with $\mu = 3, 4$.
- (c) For $f, g \in R$ suppose $ord(f) = p = ord(g)$ and present $f = f_p + f_{>p}$, $g = g_p + g_{>p}$. Prove: if $f \stackrel{\mathcal{R}}{\sim} g$ then $f_p \stackrel{GL_n(\mathbb{k})}{\sim} g_p$, where $n$ is the number of variables and the action $GL_n(\mathbb{k}) \circlearrowleft \mathbb{k}^n$ is the usual one.
- (d) Similarly prove: if $f \stackrel{\mathcal{K}}{\sim} g$ then $f_p \stackrel{GL_n(\mathbb{k})}{\sim} (\pm)g_p$.
- (e) Let $R = \mathbb{k}[[x, y]]$. Prove that the sets $x^4 + \mathfrak{m}^5$, $x^3 y + \mathfrak{m}^5$, $x^2 y^2 + \mathfrak{m}^5$ are disjoint under $\mathcal{K}$-equivalence. (i.e. no element from one set is equivalent to an element from the other)

(7)

- (a) Let $g(x, y, z) = z^3 + f(x, y)$. Express $\mu(g)$ via $\mu(f)$.
- (b) More generally, suppose $f(\underline{x}, \underline{y}) = g_1(\underline{x}) + g_2(\underline{y})$, with $\mu(g_1), \mu(g_2) < \infty$. Fix some basis of the Milnor algebra of $g_1$, as a vector space, $\{v_i^{(1)}\}$. Similarly, let $\{v_j^{(2)}\}$ be the vector space basis of the Milnor algebra of $g_2$. Prove that $\{v_i^{(1)} \otimes v_j^{(2)}\}$ is a basis for the Milnor algebra of $g_1(\underline{x}) + g_2(\underline{y})$. Express $\mu(g_1(\underline{x}) + g_2(\underline{y}))$ via $\mu(g_1(\underline{x}))$, $\mu(g_2(\underline{y}))$.