Introduction to Singularities, 201.1.0361

Homework 4

Spring 2017 (D.Kerner)

Let the field $\Bbbk$ be one of $\mathbb{R}, \mathbb{C}$, let the ring R be one of $C^\infty(\mathbb{R}^n, 0)$, $\Bbbk[[\underline{x}]]$, $\Bbbk\{\underline{x}\}$.

(1) (a) In the lecture we have defined weighted-homogeneity, $f(\lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n) = \lambda^{w_f} f(\underline{x})$. $\varnothing$
Let $f = \sum a_{\underline{m}} \underline{x}^{\underline{m}} \in \Bbbk[[\underline{x}]]$. Prove: f is weighted-homogeneous iff there exists $(w_1, \dots, w_n)$ and w_f such that for any non-zero coefficient $a_{\underline{m}}$ in f holds: $\sum w_i m_i = w_f$.
(b) Prove: if f is weighted homogeneous (possibly after a change of coordinates) then $\mu(f) = \tau(f)$.
(c) Give an example of non-isolated critical point for which $\mu(f) = \tau(f)(= \infty)$, and $f \in Jac(f)$, but f is not weighted-homogeneous in any coordinate system. (One such example was given in the lecture.)

(2) (a) Prove: $dim\, {}^{R}/_{Jac(f)} < \infty$ iff for some a holds: $dim\, {}^{R}/_{\mathfrak{m}^a \cdot Jac(f)} < \infty$.
(b) Prove: $dim\, {}^{R}/_{Jac(f)+(f)} < \infty$ iff for some a holds: $dim\, {}^{R}/_{\mathfrak{m}^a \cdot (Jac(f)+(f))} < \infty$.
(c) Prove: if $\mu(f) < \infty$ then the critical locus of f is (set-theoretically) a point. (The converse statement holds for $R = \mathbb{C}\{\underline{x}\}$), its proof will be given later.) Does the converse statement hold also for $R = \mathbb{R}\{\underline{x}\}$?

(3) (a) Prove: $x^3 + y^4 \overset{\mathcal{R}}{\sim} x^3 + y^4 + x^2y^2 + xy^3$, $x^p + y^p \overset{\mathcal{R}}{\sim} x^p + y^p + y^2x^{p-1} + x^3y^{p-1}$. (In the ring $\Bbbk\{\underline{x}\}$.)
(b) Find the order of $\mathcal{R}$-determinacy in the following cases (for $R = \Bbbk[[x, y]]$, $R = \Bbbk[[x, y, z]]$):
i. $x^3 + y^k$, ii. $x^3 + xy^3$, iii. $x^3 + y^3 + z^3$.
(c) Prove that any f with $\mu(f) = 5$ is $\mathcal{R}$-equivalent to $y^2x + x^4+$(sum of squares of the other variables). The traditional notation for this singularity is D_5.

(4) In the lecture we have defined the groups $\mathcal{R}$, $\mathcal{K}$. (Check that these are indeed groups. What is the inverse of the element $[f \to u \cdot \phi^* f]$?)
(a) Define $\mathcal{R}^{(j)} := \{g \in \mathcal{R} |\ g(\underline{x}) - \underline{x} \in \mathfrak{m}^j R^{\oplus n}\}$. Prove: $\mathcal{R}^{(j)} \lhd \mathcal{R}$ (a normal subgroup). Describe the group ${}^{\mathcal{R}}/_{\mathcal{R}^{(1)}}$.
(b) Similarly, define $\mathcal{K}^{(j)}$, prove that $\mathcal{K}^{(j)} \lhd \mathcal{K}$ (a normal subgroup), and describe ${}^{\mathcal{K}}/_{\mathcal{K}^{(1)}}$.

(5) (a) Below we assume: $R = \Bbbk[[\underline{x}]]$ and $D = \sum \phi_i \frac{\partial}{\partial x_i}$ satisfies $D(\mathfrak{m}) \subseteq \mathfrak{m}^2$. (Check that in this case: $D(\mathfrak{m}^i) \subseteq \mathfrak{m}^{i+1}$.)
(i) Prove: for any $f \in R$: $e^D(f)$ is a well defined power series, $e^D(f(\underline{x})) = f(e^D(\underline{x}))$ and $e^D(f \cdot g) = e^D(f) \cdot e^D(g)$.
(ii) Does the following identity hold: $e^D(f(\underline{x})) = f(\underline{x} + \underline{\phi})$?
(iii) Suppose $e^{D_1}(f) = e^{D_2}(f)$ for any $f \in R$. Prove that $D_1 = D_2$.
(iv) Prove that for any $f \in R$, $ln(Id + D)(f)$ is a well defined power series. Is $ln(Id + D)$ an automorphism of R?
(v) Prove: $ln(e^D) = D$ and $e^{ln(Id+D)} = Id + D$.
(b) Let D be a differential operator that contains derivatives of higher order. Do the above properties hold?
(c) Let $\{D_j\}$ be a sequence of first-order differential operators, satisfying: $D_j(\mathfrak{m}) \subseteq \mathfrak{m}^{1+j}$. Prove: that the limit $\lim\limits_{j \to \infty} (e^{D_j} e^{D_{j-1}} \cdots e^{D_1})$ exists and is a well defined automorphism of R.
(d) Let $f \in R = \Bbbk\{x\}$, with the radius of convergence r. For which $a \in \Bbbk$ is $e^{a\frac{d}{dx}}(f)$ a well defined power series?

(6) (a) Which of the following ideals (in $\Bbbk[[\underline{x}]]$) is radical? For those that are not radical, compute their radicals.
i. $(sin(sin(x)))$, ii. $(cos(x) - 1)$, iii. $(x^3 - y^5, y^3 - x^5)$.
(b) Is the ideal $\mathfrak{m}^\infty \subset C^\infty(\mathbb{R}^p, 0)$ a radical ideal?

(7) (a) Fix a domain $\mathcal{U} \subset \mathbb{C}^n$. Prove:
(i) An analytic subset of $\mathcal{U}$ is closed in $\mathcal{U}$.
(ii) A locally analytic subset of $\mathcal{U}$ is locally closed.
(iii) A locally analytic subset is analytic iff it is closed.
(b) Recall: a subset $X \subset \mathcal{U}$ is called *nowhere dense* if $Int(\overline{X}) = \varnothing$. Suppose $X \subsetneq \mathcal{U}$ is an analytic subset, prove: X is nowhere dense in $\mathcal{U}$.
(c) We have defined the notion of point-set germ as the class of equivalence under some relation. Check that this is indeed an equivalence relation.
(d) Suppose two tuples of elements define the same ideal, $(f_1, \dots, f_r) = (g_1, \dots, g_k) \subset \mathbb{C}\{\underline{x}\}$. Prove: $(V_{\underline{f}}, 0) = (V_{\underline{g}}, 0)$.