Introduction to Singularities, 201.1.0361

Homework 5

Spring 2017 (D.Kerner)

(1) (a) Check that the equivalence used in defining the germs (of sets, of functions, of maps) is indeed an equivalence relation.
(b) Verify the basic set-theoretic identities: $(X,0)\cap(Y,0)=(X\cap Y,0)$, $(X,0)\cup(Y,0)=(X\cup Y,0)$, $(X,0)\setminus(Y,0)=(X\setminus Y,0)$, $(X,0)\cap(\underset{\alpha}{\cup}(Y_\alpha,0))=\underset{\alpha}{\cup}(X\cap Y_\alpha,0)$, $(X,0)\times(Y,0)=(X\times Y,0)$.
(c) Similarly, denoting by $[f]_p$ the germ of f at a point p, check that the basic operations are well defined and there holds: $[f]_p\pm[g]_p=[f\pm g]_p$, $[f]_p[g]_p=[fg]_p$. Show that with these operations the set of germs of functions becomes a ring. Identify this ring, if one speaks of germs of analytic functions, germs of diferentiable functions.

(2) (a) Let $R=\mathbb{C}\{\underline{x}\}$, consider its ideals and analytic subgerms of $\mathbb{C}^n$. Prove:
(i) If $I_1\subset I_2$ then $(V(I_1),0)\supseteq(V(I_2),0)$. Does there hold $(V(I_1),0)\supset(V(I_2),0)$?
(ii) If $(X_1,0)\subset(X_2,0)$ then $I(X_1)\supset I(X_2)$.
(iii) $V(I(X,0))=(X,0)$. $I(V(I))\supseteq I$.
(b) Show (by examples) that Rückert's Nullstellensatz, $I(V(I))=\sqrt{I}$, does not hold in the rings $\mathbb{R}\{\underline{x}\}$, $C^\infty(\mathbb{R}^p,0)$.

(3) (a) Which of the following rings are Noetherian?
i. $C^\infty(\mathbb{R}^p,0)$, ii. $\Bbbk[x_1,x_2,\dots]$, iii. $\mathcal{O}(\mathcal{U})$ for some (open) domain $\mathcal{U}\subset\mathbb{C}$. (Here take any sequence $\{x_n\}$ with no condensation points inside $\mathcal{U}$. Consider $I_n=\{f\in\mathcal{O}(\mathcal{U})|\ f(x_n)=f(x_{n+1})\}=\dots=0$.
(b) Suppose R is Noetherian. Prove that for any system of generators, $\{f_\alpha\}_\alpha$ of an ideal $I\subset R$ there exists a finite generating subsystem.

(4) (a) Show (by examples) that $C^\infty(\mathbb{R}^p,0)$ is not a domain.
(b) Show (by examples) that $\Bbbk\{x,y\}/(y^2-x^3)$ is not a unique factorization domain.
(c) Prove: any subring of $\Bbbk[[\underline{x}]]$ is a domain. (In particular the rings $\Bbbk[\underline{x}]$, $\Bbbk\{\underline{x}\}$.)
(d) Prove: the analytic set-germ $(X,0)\subset(\mathbb{C}^n,0)$ is irreducible iff the ideal $I(X,0)\subset\mathbb{C}\{\underline{x}\}$ is prime.

(5) Compute the codimensions of the following sets: $\{\sum\limits_{j=1}^n x_j^3=0\}\subset\mathbb{C}^3$, $\{xz=y^2,\ x^3=z^5,\ y^3=z^4\}\subset\mathbb{C}^3$.

(6) Given an analytic map $(\mathbb{C},0)\to(\mathbb{C}^2,0)$, $t\to(x(t),y(t))$. Prove: this is a parametrization of the curve germ $(C,0)=\{f(x,y)=0\}$ iff $f(x(t),y(t))\equiv0$ and the subring $\mathbb{C}\{x(t),y(t)\}\subseteq\mathbb{C}\{t\}$ contains the ideal (t^N) for some $N\gg1$.

(7) In this question $R=\Bbbk[[\underline{x}]]$.
(a) Suppose $\Bbbk=\bar{\Bbbk}$ and $f\in R$ is weighted homogeneous. Prove the factorization $f(x,y)=\prod(a_iy^p-b_ix^q)$, for some $p,q\in\mathbb{N}$ and $\{a_i\},\{b_i\}\in\Bbbk$. Draw Γ_f.
(b) Draw the Newton diagram of $(y^{p_1}+x^{q_1})(y^{p_2}+x^{q_2})$, where $\frac{q_1}{p_1}<\frac{q_2}{p_2}$.
(c) Prove: for any $f\in R$ the diagram Γ_f depends on (f) only, i.e. $\Gamma_f=\Gamma_{uf}$ for any invertible $u\in R$.
(d) Here we assume $\Bbbk=\bar{\Bbbk}$. Fix a face $\sigma\subset\Gamma_f$. Prove: $f|_\sigma=x^my^n\prod(a_iy^p-b_ix^q)$, for some $m,n,p,q\in\mathbb{N}$ and $\{a_i\},\{b_i\}\in\Bbbk$.

(8) (a) Write down (explicitly) the Newton-Puiseux parametrization for the curve singularity $\{y^3-x^5-3x^4y-x^7=0\}$.
(b) Write the first few terms of the parametrization for $\{y^p=x^q+yx^{q-1}\}$, here $p<q$, $gcd(p,q)=1$.
(c) Find the curve singularity (the equation) whose parametrization is:
i. (t^3,t^2+t^4), ii. (t^6,t^8+t^{13}).
(d) Show that the germ $\{x^5-x^2y^2+y^5=0\}$ has two branches (draw the Newton diagram for each of them) and find the first two terms of the Puiseux series for each of them.
(e) Fix a field $\Bbbk$, of zero characteristic, and a branch $\{f(x,y)=0\}\subset(\Bbbk^2,0)$. Given a parametrization $(x(t),y(t))\in\Bbbk[[t]]$, with $x(t)$ monic of order p, prove: there exists a parametrization $t\to(t^p,\tilde{y}(t))$.

(9) (a) Suppose the germ $(C,0)\subset(\mathbb{C}^2,0)$ has multiplicity two. Prove that in some local coordinates this germ is: $\{y^2=x^n\}$.
(b) Suppose the germ $(C,0)\subset(\mathbb{C}^2,0)$ consists of a germ $(\tilde{C},0)$ of multiplicity two and a smooth germ $(l,0)$ non-tangent to $(\tilde{C},0)$. Prove that in some local coordinates $(C,0)=\{x(y^2+x^n)=0\}$.