Introduction to Singularities, 201.1.0361

Homework 6

Spring 2017 (D.Kerner)

(1) (a) In our proof of the existence of parametrization we have constructed (inductively) the map $\begin{matrix}(\mathbb{C},0)\to(C,0)\\ t\to(x(t),y(t))\end{matrix}$.
We did not have time to prove two properties: i. this map is injective, ii. $ord_t(x(t)) = mult(C,0)$. Prove them.

(b) Given a parametrization $\begin{matrix}(\mathbb{C},0)\to(C,0)\\ t\to(x(t),y(t))\end{matrix}$, with $ord_t x(t) = p$, construct a parametrization $\tilde{t}\to(\tilde{t}^p, y(\tilde{t}))$.

(c) Suppose $(C,0)$ is smooth with a parametrization $(\mathbb{C},0)\overset{\phi}{\to}(C,0)$. Rectify $(C,0)$ to $\{y=0\}\subset(\mathbb{C}^2,0)$. Prove that the corresponding map $(\mathbb{C},0)\overset{\phi}{\to}\{y=0\}$ is an analytic isomorphism. (i.e. both ϕ and ϕ^{-1} are analytic)

(2) (a) Consider the set $\cup_{m\ge1}\mathbb{C}\{x^{\frac{1}{m}}\}$, here every element is a *finite* sum $\sum a_m(x^{\frac{1}{m}})$, where $a_m(t)\in\mathbb{C}\{t\}$. Prove that this set is a local ring. (The name: the ring of Puiseux power series.) Prove that in this ring the implicit function theorem holds.

(b) The existence of parametrization implies: for any ($\hat{y}$-general) series $f(x,y)\in\mathbb{C}\{x,y\}$, $f(0,0)=0$, the equation $f=0$ has a solution $y(x)\in\cup_{m\ge1}\mathbb{C}\{x^{\frac{1}{m}}\}$. Prove a stronger property: for any ($\hat{y}$-general) series $f(x,y)\in\Big(\cup_{m\ge1}\mathbb{C}\{x^{\frac{1}{m}}\}\Big)\{y\}$, (i.e. power series in y, whose coefficients are series in fractional powers of x), with $f(0,0)=0$, the equation $f=0$ has a solution $y(x)\in\cup_{m\ge1}\mathbb{C}\{x^{\frac{1}{m}}\}$.

(3) Let $f,g\in\mathbb{C}\{x,y\}$.

(a) Show that $i(f,g)\ge mult(f)\cdot mult(g)$.

(b) Prove: $mult(f) = min\{i(f,g)|\ g\in\mathfrak{m}\subset\mathbb{C}\{x,y\}\}$. Prove that the minimum is attained for $g(x,y)=ax+by$, with a,b generic.

(4) (a) Prove that a line l is tangent to $\{f=0\}$ iff $i(f,l)>mult(f)$. In particular, if f is irreducible, prove: $mult(f) = min\Big(ord(x(t)), ord(y(t))\Big)$. (Here $(x(t),y(t))$ is a parametrization.) When is the tangent cone a linear space?

(b) Define the order of tangency of smooth germs $(C_1,0)$ $(C_2,0)$ as $i(C_1,C_2)$. Prove: if this order equals $p\ge1$, then in some coordinates holds: $(C_1,0)=\{y=0\}$, $(C_2,0)=\{y=x^p\}$.

(c) Fix a branch $(C,0)\subset(\mathbb{C}^2,0)$. For any two points $p,q\in C$ take the line $\overline{p,q}\subset\mathbb{C}^2$. Fix some sequences $\{p_n\},\{q_n\}$ converging to $(0,0)\in\mathbb{C}^2$. Suppose $\lim\overline{p_n,q_n}$ exists. Is this necesarily a tangent line of $(C,0)$?

(d) Prove that the set of tangent lines of $\{f_p+f_{>p}=0\}$ is defined by the (reduced) linear factors of $f_p(x,y)$.

(e) Suppose the Newton diagram, $\Gamma_{(C,0)}$ is convenient. Its integral lenght is defined as the number of $\mathbb{Z}^2$ points on $\Gamma_{(C,0)}$, minus one. Prove: the number of tangent lines (counted without multiplicities) is at most the integral length of $\Gamma_{(C,0)}$. When does the inequality occur? Does the bound hold when the tangent lines are counted with multiplicities?

(5) (a) Consider the curve germ $(C,0)=\{\prod\limits_{\substack{i=1\dots r\\ \alpha_i\neq\alpha_j}}(y^p-\alpha_i x^q)=0\}$. Apply the generic coordinate translations to all the branches, to reach a curve whose components intersect only at smooth points and each such intersection point is a node. How many nodes you get?

(b) Fix two smooth real germs, $(C_1,0),(C_2,0)\subset(\mathbb{R}^2,0)$,with $i(C_1,C_2)=p>1$. Can you deomnstrate a *real* deformation, $C_1(\epsilon)$, $C_2(\epsilon)$, that splits the intersection point at $(0,0)$ into p nodes? (Hint: fix p points on the $\hat{x}$-axis and force $C_1(\epsilon)$, $C_2(\epsilon)$ to pass through them.)

(6) (a) Given an isolated singularity $(C,0)$, fix some $Ball_\delta(0)$ and a deformation C_ϵ of C. Does there hold, for $|\epsilon|\ll1$, $\mu(C,0)=\sum\limits_{pt\in C_\epsilon\cap Ball_\delta(0)}\mu(C_\epsilon,pt)$? (What is the difference between this formula and the one given in the class?)

(b) Let $f_1(x,y), f_2(x,y)$ be homogeneous polynomials of degrees d_1,d_2, with no common factors. Compute $i(f_1,f_2)$.

(c) Let $f(x,y)$ be a homogeneous polynomial of degree p, with no multiple factors. Prove: $\mu(f)=(p-1)^2$.

(d) Fix a (not necessarily homogeneous) polynomial $f(x,y)$, of degree p. Suppose the curve $C=f^{-1}(0)\subset\mathbb{C}^2$ has an isolated singularity at the origin. Prove: $\mu(C,0)\le(p-1)^2$. (Hint: choose the generic coordinate axes, such that $f(x,y)$ contains the monomials x^p, y^p. Present f as a deformation of some homogeneous polynomial of degree p.)