Introduction to Singularities, 201.1.0361

Homework 7

Spring 2017 (D.Kerner)

(1) Getting used to the blowup $Bl_0(\mathbb{C}^2) \to \mathbb{C}^2$. In the last lecture(s) we spoke about the blowup of the complex plane. The following is meant to help to get used to this notion.

- (a) Cover the complex projective line, $\mathbb{P}^1_{\mathbb{C}}$, by two charts, each isomorphic to $\mathbb{C}^1$. Fix the coordinates and write down the transition maps between the coordinates of the charts. (What is $\mathbb{P}^1_{\mathbb{C}}$ topologically?)
- (b) Define the real projective line, $\mathbb{P}^1_{\mathbb{R}}$, similarly to $\mathbb{P}^1_{\mathbb{C}}$. Repeat the previous steps. (What is $\mathbb{P}^1_{\mathbb{R}}$ topologically?)
- (c) Going along the same lines as on the lecture, define the real blowup, $Bl_0(\mathbb{R}^2) \stackrel{\pi}{\to} \mathbb{R}^2$. Realize $Bl_0(\mathbb{R}^2)$ as a subset in some real three-dimensional manifold (which one?) Cover $Bl_0(\mathbb{R}^2)$ by two charts, each isomorphic to $\mathbb{R}^2$. Write down the transition functions. Prove that $Bl_0(\mathbb{R}^2)$ is a smooth (connected) real surface. Prove that the exceptional locus of π is a compact connected real manifold (without boundary) of dimension one. What is this manifold?
  Note: in both coordinate charts the defining equation of $Bl_0(\mathbb{R}^2)$ resembles the one of the saddle point of calculus, e.g. $y = x\frac{\sigma_y}{\sigma_x}$. Use this to visualize $Bl_0(\mathbb{R}^2)$.
- (d) Given $Bl_0(\mathbb{C}^2) \stackrel{\pi}{\to} \mathbb{C}^2$, prove that $\pi^{-1}(0)$ is a smooth compact complex curve. (Over reals this is a compact two-dimensional manifold, without boundary. What is this manifold?)
- (e) Realize $Bl_0(\mathbb{C}^2)$ as the closure of the graph of the map $\mathbb{C}^2 \setminus \{(0,0)\} \stackrel{f}{\to} \mathbb{P}^1$, $f(x,y) = (x:y)$. Do the same for $Bl_0(\mathbb{R}^2)$.
- (f) Let l_1, l_2 two (distinct) lines through the origin of $\mathbb{R}^2$ or $\mathbb{C}^2$. What are their strict transforms, $\tilde{l}_1$, $\tilde{l}_2$? Compare the intersection multiplicities $i(l_1, l_2)$, $i(\tilde{l}_1, \tilde{l}_2)$.
- (g) See also: *https://en.wikipedia.org/wiki/Blowing_up.*

(2)
- (a) Consider the strict transform of the curve singularity $(C,0) = \{x^p = y^p\} \subset (\mathbb{C}^2, 0)$ under the blowup $Bl_0(\mathbb{C}^2) \stackrel{\pi}{\to} \mathbb{C}^2$. Write down the defining equation(s) of the strict transform.
- (b) Prove that an ordinary multiple point is resolvable by one blowup. (Namely, if the curve germ $(C,0)$ consists of several smooth branches, pairwise non-tangent, then the strict transform under one blowup is a collection of smooth curve-germs, intersecting the exceptional divisor transversally.)
- (c) Prove: the strict transform of the union of curves is the union of the strict transforms, $\widetilde{(\cup C_i, 0)} = \cup(\tilde{C}_i, 0)$.
- (d) Given a branch $(C,0)$, with tangent $\hat{y}$, and the parametrization $(x(t), y(t))$. Prove that the parametrization of $\tilde{C}$ is $(x(t), \frac{y(t)}{x(t)})$.
- (e) Given two smooth curve germs, $(C_1,0), (C_2,0) \subset (\mathbb{C}^2, 0)$, with tangency of order k, i.e. $i_0(C_1, C_2) = k$. What is the minimal number of blowups needed to separate these curves?

(3) It was stated in the lecture that $Bl_0(\mathbb{C}^2) \stackrel{\pi}{\to} \mathbb{C}^2$ does not depend on the local coordinates in $(\mathbb{C}^2, 0)$, up to isomorphism. More precisely, the statement is: any isomorphism of germs of complex smooth surfaces, $(\mathcal{U}_1, pt_1) \stackrel{\phi}{\tilde{\to}} (\mathcal{U}_2, pt_2)$, lifts to an isomorphisms of the blowups. Namely, the diagram on the right commutes. Formulate the similar proposition over $\mathbb{R}$ and prove it.

$$\begin{array}{ccc} Bl_{pt_1}(\mathcal{U}_1, pt_1) & \tilde{\to} & Bl_{pt_2}(\mathcal{U}_2, pt_2) \\ \downarrow & & \downarrow \\ (\mathcal{U}_1, pt_1) & \stackrel{\phi}{\tilde{\to}} & (\mathcal{U}_2, pt_2) \end{array}$$

(4)
- (a) Draw the embedded resolution of the singularities $y^p = x^p$, $y^p = x^{pk}$, $y^p = x^{p+1}$.
- (b) Resolve the singularity $y^5 + y^2x^2 + x^5 = 0$ and compute its Milnor number.