Algebra, Geometry and Topology of Singularities

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Schedule

- (9:00-9:20) Cofee&Cookies
- (9:20-10:05) E. Shustin, "Singular Welschinger invariants".
- (10:15-11:00) K. Christ, "On irreducibility of Severi varieties of toric surfaces".
- (11:10-11:55) R. Tessler, "Deformation of A_n singularities and $(n+1)$-spin curves".
- (12:05-12:50) R. Mendes Pereira, "On the Link of Lipschitz Normally embedded singularities".
- (13:00-14:30) Lunch
- (14:30-15:15) V. Rovenski, "Integral formulas for singular distributions".
- (15:25-16:10) Y. Yomdin,"Singularity Theory in Super-resolution problem".
- (16:20-17:05) G.M. Greuel "On Semicontinuity of Singularity Invariants in Families of Formal Power Series".

Abstracts

(1) E. Shustin, *Singular Welschinger invariants.*
A versal deformation of a real plane curve singularity contains the so-called Severi loci parameterizing deformations with a given total delta-invariant (and among them the equigeneric stratum), and the cuspidal equigeneric loci (among them the equiclassical stratum). The multiplicities of these analytic spaces germs can be regarded as local analogues of Gromov-Witten invariants. We show that some of them possess real multiplicities, which can be viewed as local analogues of Welschinger invariants. We also show that some of these real invariants can be converted to global invariants that count real rational curves having prescribed number of cusps and belonging to a suitable linear system on a real toric surface.

(2) K. Christ, *On irreducibility of Severi varieties of toric surfaces.*
Severi varieties parametrize irreducible curves of fixed geometric genus in a given linear system. They are classical objects that have been studied extensively. My talk will survey results concerning the irreducibility of such varieties for linear systems defined on toric surfaces. The case of the projective plane in characteristic zero was settled in the landmark paper of Harris in 1986. After recalling his approach, I will sketch further developments since then. Finally, I will report on ongoing work with Xiang He and Ilya Tyomkin, in which we use tropical geometry to study the question of irreducibility. In particular, we can prove irreducibility for the projective plane in positive characteristic. Time permitting, I will indicate some further directions we plan to investigate.

(3) R. Tessler, *Deformation of A_n singularities and $(n+1)$-spin curves.*
I will review Witten's r-spin conjecture (Faber-Shadrin-Zvonkin's theorem) which relates the potential of Gromov-Witten-like invariants of r-spin curves with the Gelfand-Dickey hierarchy. I will then review its recent open analog, which concerns r-spin surfaces with boundaries. I will finish by showing how the the open genus 0 invariants appear in the theory of versal deformations of A_{r-1} singularities. Based on joint works with Buryak-Clader and with Gross-Kelly.

(4) R. Mendes-Pereira, *On the link of Lipschitz normally embedded singularities.*
A semialgebraic germ $(X, x_0) \subset (\mathbb{R}^n, x_0)$ has two natural metrics: the outer (or euclidean) metric and the inner metric (or length metric). When the two metrics are bi-Lipschitz equivalent and the bi-Lipschitz homeomorphism is given by the identity map, we say that the germ (X, x_0) is Lipschitz normally embedded. This notion, introduced in this way by Birbrair and Mostowski, is in a recent development and enables one to understand the nature of the singular types of algebraic varieties on the metric point of view. In this talk, we prove that a germ (X, x_0) is Lipschitz normally embedded if, and only if, the family of ϵ-Links (for small ϵ) $\{X \cap \mathbb{S}^{n-1}(x_0, \epsilon)\}_\epsilon$ is Lipschitz normally embedded with uniform constant. In the last part (considering the time), we will discuss about the behaviour of the some metric invariants of singular germ using this equivalence.
This a joint work with Jose Edson Sampaio.

(5) V. Rovenski, *Integral formulas for singular distributions.*

Distributions, being subbundles of the tangent bundle on a manifold, and *foliations*, i.e., integrable distributions, arise in many topics of mathematics and its applications [1, 2]. The mixed curvature is a basic concept of differential geometry of distributions and foliations (a plane, which intersects nontrivially both distributions, is called mixed). Studying of singularities is important for the theory of foliations: a manifold may admit no smooth codimension-one distributions or foliations, while it admits such ones defined outside some "set of singularities". We deal with two concepts of singularity.

i) A *distribution with singularities* is one defined on a manifold outside a finite union, Σ, of closed submanifolds of codimension $k \geq 2$. In this case, we obtain integral formulas with the mixed curvature, using improper integrals: if $(k-1)(p-1) \geq 1$ and β is a $(\dim M - 1)$-form on $M \setminus \Sigma$ with Riemannian metric g such that $\int_M \|\beta\|^p \,\mathrm{d}V_g < \infty$, then $\int_M d\beta = 0$.

ii) *Singular distributions* (i.e., of varying dimension) can be defined as images (or parameterizations) of the tangent bundle by smooth endomorphisms of varying rank. For a singular distribution we prove the modified divergence theorem, and for a pair of such distributions we prove the Codazzi type equation. Tracing this equation and applying our divergence theorem, we get the integral formula with the mixed scalar curvature.

References

[1] A. Candel and L. Conlon, *Foliations, I and II*, Grad. Studies in Math. 60, AMS, 2000, 2003.
[2] A. Isidori, *Nonlinear Control Systems*, 3-rd ed., Springer Verlag, 1995.
[3] P. Popescu and V. Rovenski, An integral formula for a pair of singular distributions, preprint, arxiv:1908.07261 [DG], 2019.

(6) Y. Yomdin, *Singularities of Super-Resolution problem*

Super-resolution problem (in one of its numerous appearances) is as follows: we consider Fourier Reconstruction of spike-train signals, i.e. of linear combinations of delta-functions

$$F(x) = \sum_{j=1}^{d} a_j \delta(x - x_j).$$

In an important case when some of the nodes $x_1, \ldots, x_d$ nearly collide, while the measurements are noisy, a dramatic error amplification may occur in the process of reconstruction. At least in part, this error amplification reflects the geometric nature of the problem itself, and does not depend on the choice of the solution method. The problem is to analyse the geometry of the "measurement error amplification", and, ultimately, to suggest mathematical ways to improve reconstruction accuracy.

This problem is known to present a remarkable variety of mathematical phenomena, from interpolation of entire functions, to cancellation in trigonometric sums, to complicated and highly singular Algebraic Geometry, and, of course, to Singularity Theory. We plan to discuss some results and open questions in Super-Resolution problem, from the point of view of Singularity Theory.

(7) G.M. Greuel *On Semicontinuity of Singularity Invariants in Families of Formal Power Series.*

I report on work in progress together with Gerhard Pfister.

We consider families of finitely presented modules M, where the entries of the presentation matrix are formal power series, parametrized by the spectrum of an arbitrary Noetherian ring A. We assume that the fibre over some prime ideal P in A has finite dimension $d_P(M)$, as a vector space over the residue field $k(P)$. We say then that M is quasifinite over P. In such a situation the fibres are defined by formal power series while the basis may be an affine algebraic variety. Such families appear naturally when one classifies, say, isolated singularities, and a natural question is, whether the Milnor number is semicontinuous under deformations.

In contrast to the complex analytic situation, where "quasifinite" implies "finite" and where the semicontinuity of $x \to d_x(M)$ is well known, in our situation the semicontinuity of $P \to d_P(M)$ is a surprisingly subtle problem. In fact, semicontinuity is in general wrong on $Spec(A)$ even for $A = K[t]$, but holds under certain assumptions on $Max(A)$, the set of maximal ideals (the closed points of $Spec(A)$). The situation becomes better if we consider the completed fibres (which we introduce) and I will talk on the results obtained so far. Finally I apply this to several invariants of singularities.