LINEAR ALGEBRA FOR MECHANICAL ENGINEERING, SKETCHY SOLUTIONS OF THE MIDTERM, (09.06.2016)

(1) (a) Present the expression in the form $\frac{\left(\frac{2}{\sqrt{3}}\left(cos(-\frac{\pi}{3})+i\cdot sin(-\frac{\pi}{3})\right)\right)^{24}}{\left(\frac{2}{\sqrt{3}}\left(cos(-\frac{\pi}{6})+i\cdot sin(-\frac{\pi}{6})\right)\right)^{36}}$ and use the formula of de Moivre to get: $(\frac{2}{\sqrt{3}})^{-12}\frac{\left(cos(-8\pi)+i\cdot sin(-8\pi)\right)}{\left(cos(-6\pi)+i\cdot sin(-6\pi)\right)}$. And the later expression equals $(\frac{3}{4})^6$.

(b) Using $sin(\alpha+\beta)=sin(\alpha)cos(\beta)+cos(\alpha)sin(\beta)$ one gets: $Span_{\mathbb{R}}\big(sin(x+1),cos(x-1)\big)\ni sin(x),cos(x)$. Therefore $Span_{\mathbb{R}}\big(cos(x+ln(2)),sin(x+\sqrt{2}),sin(x+1),cos(x-1)\big)=Span_R\big(sin(x),cos(x)\big)$. Thus the needed dimension is 2.

(2) (a) a. Bring the system to the canonical form. In terms of matrices after a few steps one has:

$$\left[\begin{array}{ccc|c}1&1&1&1\\a&b&c&b\\a^2&b^2&c^2&a^2\end{array}\right]\rightsquigarrow\left[\begin{array}{ccc|c}1&1&1&1\\0&b-a&c-a&b-a\\0&0&(c-a)(c-b)&a^2-b^2\end{array}\right]$$

Thus:

- if a,b,c are all distinct then there exist the unique solution.
- if ($c=a$ or $c=b$) but $a^2\neq b^2$ then there is no solution.
- if $a=b$ then there are infinity of solutions.
- if $-b=a\neq b$ and $(c-a)(c-b)=0$ then there is infinity of solutions.

(b) i. Note that the condition $C^2AB^{-2}=BA^tC$ is linear in A. (It is satisfied by the zero matrix. If A satisfies it then $\lambda\cdot A$ satisfies as well. If A, $\tilde{A}$ satisfy it then $A+\tilde{A}$ satisfies it as well.) Therefore V is a vector subspace of $M_{3\times3}(\mathbb{R})$.

ii. If V is a vector subspace, then it must contain the zero vector of the ambient space. (Which is the "zero" sequence: $0,0,0,\dots$.) But the sequence of zeros does not converge to 1. Therefore V does not contain the sequence of zeros, thus V is not a vector subspace.

(3) (a) Denote the columns of A by C_1,C_2,C_3. Then $M^{up}_{3\times3}(\mathbb{C})\cap V$ is the vector space of all the upper triangular matrices whose columns satisfy: $C_1+2C_2+3C_3=0$. Therefore the equations defining $M^{up}_{3\times3}(\mathbb{C})\cap V$ are: $a_{33}=0$, $2a_{22}+3a_{23}=0$, $a_{11}+2a_{12}+3a_{13}=0$. Thus a basis for $M^{up}_{3\times3}(\mathbb{C})\cap V$ is:

$$\begin{bmatrix}-2&1&0\\0&0&0\\0&0&0\end{bmatrix},\quad\begin{bmatrix}-3&0&1\\0&0&0\\0&0&0\end{bmatrix},\quad\begin{bmatrix}0&0&0\\0&-3&2\\0&0&0\end{bmatrix}.$$

(b) The assumptions imply: $Span_{\mathbb{R}}(L_1,L_2)\ni(2,1),(1,2)$. The later vectors are linearly independent, thus $dimSpan_{\mathbb{R}}(L_1,L_2)\geq2$. Therefore the rows L_1,L_2 span the vector space $\mathbb{R}^2$. Thus (by the theorem of equivalent conditions for invertibility) A is invertible.

(4) (a) If $det(A)\neq0$ then the canonical form of A is the unit matrix, $\mathbb{1}$. And the same for A^t. Thus both A and A^t are row-equivalent to the same matrix, hence they are row-equivalent.

(b) Let $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ then $A^t=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ and the two matrices are not row-equivalent. (e.g. because $Span(Row(A))=Span(0,1)$ but $Span(Row(A^t))=Span(1,0)$.)