LINEAR ALGEBRA FOR MECHANICAL ENGINEERING,
SKETCHY SOLUTIONS OF MOED.B, (9.08.2016)

(1) (a) *Solution 1.* The geometric meaning of $|z-1+i|$ is the distance between the points z, $1-i$. Thus the condition $|z-1+i| = |z+1-i|$ means: the point $z = x+iy$ is equidistant to the points $(1-i)$ and $(i-1)$. Or, in terms of $\mathbb{R}^2$, the point (x,y) is equidistant to the points $(1,-1)$, $(-1,1)$. But this means: z lies on the line $\{x=y\} \subset \mathbb{R}^2$. All the points of this line satisfy this equation.

Solution 2. Square both parts and use $|z|^2 = z\bar{z}$ to get: $(z-1+i)(\bar{z}-1-i) = (z+1-i)(\bar{z}+1+i)$. Open the brackets and move the right hand side to the left hand side to get: $2(\bar{z}(i-1)+z(-1-i)) = 0$. Present $z = x+iy$, then the condition is: $y = x$.

(b) Let $A = B = \mathbb{O}$ then the condition means: $(-\lambda)^n = (-\lambda)^n(-\lambda)^n$. And this does not hold for $\lambda \neq \pm 1$.

(2) (a) The condition $p(-x) = p(x)$ means that any element of V is an even function, thus it contains only the even powers of x. Therefore $V \subseteq Span(1, x^2, x^4, \dots, x^{10})$.
The condition $p(0) = 0$ means that the free coefficient of this polynomial vanishes. The condition $p'(0) = 0$ is then satisfied. Therefore a basis for V is $(x^2, x^4, x^6, x^8, x^{10})$, in particular $dim(V) = 5$. Thus a complementary subspace can be chosen as $Span(1, x, x^3, x^5, x^7, x^9)$, and these vectors form its basis.

(b) Use the presentation $\hat{x} = (1,0) = \frac{2(2,1)-(1,2)}{3}, \hat{y} = (0,1) = \frac{-(2,1)+2(1,2)}{3}$ to get:

$$< \hat{x}, \hat{y} > = < \frac{2(2,1)-(1,2)}{3}, \frac{-(2,1)+2(1,2)}{3} > = -\frac{4}{9}.$$

(3) (a) As V is finite dimensional we can use the comparison of dimensions: $dim(Im(T)) + dim(ker(T)) = dim(V)$. As $Im(T) = V$ we get $dim(ker(T)) = 0$, i.e. $ker(T) = \{0\}$. Therefore T is injective.

(b) As T is surjective and injective, it is invertible. Thus $T^2 = T$ implies: $T = T^{-1} \circ T^2 = T^{-1} \circ T = Id$.

(4) (a) Note that $T(1,0) = (0,-1)$ and $T(0,1) = (-1,0)$, thus $[T]_E = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. The presentation matrix of the projection is $[S]_E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore $[S \circ T]_E = [S]_E[T]_E = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$.

(b) The characteristic polynomial of $[S \circ T]_E$ is: $det(x\mathbb{1} - [S \circ T]_E) = x^2$. Thus the only eigenvalue is $\lambda = 0$, its algebraic multiplicity is 2. The corresponding eigenvectors are the solutions of the system $[S \circ T]_E v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The space of the solutions is spanned by just one vector, $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so the geometric multiplicity is one. In particular the geometric multiplicity is smaller than the algebraic multiplicity. Therefore the map $S \circ T$ is non-diagonalizable.