LINEAR ALGEBRA FOR MECHANICAL ENGINEERING, SKETCHY SOLUTIONS OF MOED.C, (15.09.2016)

(1) (a) Present the initial expression as follows: $\sqrt{1-i\sqrt{3}} = \sqrt{2}\sqrt{cos(-\frac{\pi}{3}) + i\cdot sin(-\frac{\pi}{3})}$.
By de Moivre formula we have: $\Big(cos(-\frac{\pi}{6}) + i\cdot sin(-\frac{\pi}{6})\Big)^2 = cos(-\frac{\pi}{3}) + i\cdot sin(-\frac{\pi}{3})$. Therefore

$$\sqrt{cos(-\frac{\pi}{3}) + i\cdot sin(-\frac{\pi}{3})} = \pm\Big(cos(-\frac{\pi}{6}) + i\cdot sin(-\frac{\pi}{6})\Big).$$

Therefore $\sqrt{1-i\sqrt{3}} = \pm\frac{\sqrt{3}-i}{\sqrt{2}}$.

(b) Apply Gauß algorithm to bring A to the canonical form, $A \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore $ker(A)$ is spanned by $(1,-2,1)$. Thus $dim\Big(ker(T_A)\cap Im(T_A)\Big) \le 1$, and this dimension is 1 if and only if the kernel vector $(1,-2,1)$ is a linear combination of the columns of A. One way to check this is to consider the "extended" matrix $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & -2 \\ 0 & -3 & -6 & 1 \end{bmatrix}$ and to check its column rank. By the direct check, the rank of this matrix is three, thus $(1,-2,1)$ is linearly independent of the columns of A. Thus $ker(T_A)\cap Im(T_A) = \{0\}$, hence $dim\Big(ker(T_A)\cap Im(T_A)\Big) = 0$.

(2) (a) We have: $T\Big(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\Big) = \begin{bmatrix} a_{11}-a_{22} & 0 \\ 0 & a_{22}-a_{11} \end{bmatrix}$. Therefore

$$ker(T) = \Big\{\begin{bmatrix} a & a_{12} \\ a_{21} & a \end{bmatrix} \mid a, a_{12}, a_{21} \in \mathbb{C}\Big\}, \quad Im(T) = Span\Big(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\Big).$$

(b) By definition, $T|_{ker(T)} = \mathbb{O}$, i.e. $ker(T)$ is an eigenspace. Furthermore, for any $A \in Im(T)$ one has: $T(A) = 2A$. Thus $Im(T)$ is an eigenspace that corresponds to the eigenvalue 2. Finally, $M_{2\times 2}(\mathbb{C}) = ker(T)\oplus Im(T)$. Therefore T is diagonalizable. An example of diagonalizing basis is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The presentation matrix of T in this basis T is: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(3) By the assumptions we have: $dim(V_1) = 1 = dim(V_2)$. Therefore $dim(V_1\cap V_2) \le 1$, and the inequality can be strict. For example, let $V = Span(\hat{x}_1, \hat{x}_2, \hat{x}_3)$, $V_1 = Span(0,0,0,1)$, while $V_2 = Span(1,0,0,1)$. Then $V_1\cap V_2 = \{0\}$.

Finally, we use the theorem on dimensions, $dim(V+W) = dim(V) + dim(W) - dim(V\cap W)$, to get: $dim(V_1 + V_2) \le 2$.

(4) (a) By its definition $V = Span(1, x^3)$. Therefore, to find an orthonormal basis, it is enough to apply Gram-Schmidt process to the vectors $1, x^3 \in V$. Note that $\int_{-1}^{1} 1\cdot x^3 dx = 0$, i.e. the vectors are already orthogonal. Thus it is enough to normalize them. As $\int_{-1}^{1} 1 dx = 2$, $\int_{-1}^{1} x^3 dx = \frac{2}{7}$, the orthonormal basis of V is: $v_1 = \frac{1}{\sqrt{2}}$, $v_2 = \frac{x^3}{\sqrt{2/7}}$.

(b) By its definition $W = \Big\{\begin{bmatrix} a & a_{12} \\ a_{21} & -a \end{bmatrix} \mid a, a_{12}, a_{21} \in \mathbb{R}\Big\}$. Thus $dim(W) = 3$ and $dim(W^\perp) = 1$, i.e. $W^\perp$ is spanned by just one vector. To find this vector one can, e.g. fix a simple basis of W:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Suppose $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in W^\perp$, then the conditions $trace(A_1B^t) = 0 = trace(A_2B^t) = trace(A_3B^t)$ imply: $b_{12} = 0 = b_{21} = b_{11} - b_{22}$. Therefore B is a multiple of the unit matrix. Thus a basis for $W^\perp$ is $\mathbb{1}_{2\times 2}$.