INTRODUCTION TO TOPOLOGY,
SKETCHY SOLUTIONS OF THE MIDTERM, (09.06.2016)

(1) (a) The spaces are not homeomorphic.

Suppose X, Y are homeomorphic, fix some homeomorphism $X \overset{f}{\xrightarrow{\sim}} Y$. Then $X \setminus \{(0,0,0)\} \approx f(X \setminus \{0,0,0\}) = Y \setminus \{pt\}$ for some point $pt \in Y$. But $X \setminus \{0,0,0\})$ is non-connected (e.g. it is separated into two parts by the plane $z = 0$), while $Y \setminus \{pt\}$ is path-connected, for any $pt \in Y$.

(b) The spaces are not homeomorphic. Indeed, Y is compact (being closed and bounded inside $\mathbb{R}^n$), while X is non-compact (it is not-closed in $\mathbb{R}^n$).

(2) (a) The sequence $\{x_i^{(n)}\}_n$ converges to $(0,0,\dots)$ in $\mathcal{T}^{uniform}_{\prod\limits_{i=1}^{\infty}\mathbb{R}}$. Indeed, take any basic neighborhood of $(0,0,\dots)$, i.e. an open ball $Ball_\epsilon\big((0,0,\dots)\big)$. Fix some N such that $\frac{1}{N^N} < \epsilon$, then $\frac{1}{(i\cdot n)^{i\cdot n}} < \epsilon$ for any $i \geq 1$ and $n \geq N$. Thus all the elements $\{x_i^{(n)}\}_n$, for $n > N$, belong to the ball.
As $\mathcal{T}^{uniform}_{\prod\limits_{i=1}^{\infty}\mathbb{R}} \supseteq \mathcal{T}^{product}_{\prod\limits_{i=1}^{\infty}\mathbb{R}}$ we get: the sequence $\{x_i^{(n)}\}_n$ converges to $(0,0,\dots)$ also in the product topology.
However the sequence does not converge to any point in $\mathcal{T}^{box}_{\prod\limits_{i=1}^{\infty}\mathbb{R}}$. Indeed, the convergence of the sequence implies the convergence of all its "coordinate-sequences". Thus the sequence could converge only to the point $(0,0,\dots)$. Thus it is enough to demonstrate an open neighborhood of $(0,0,\dots)$ that does not contain infinity of elements of $\{x^{(n)}\}_{i,n}$. Take e.g. the neighborhood

$$(-\frac{1}{(1!)^{1!}}, \frac{1}{(1!)^{1!}}) \times (-\frac{1}{(2!)^{2!}}, \frac{1}{(2!)^{2!}}) \times (-\frac{1}{(3!)^{3!}}, \frac{1}{(3!)^{3!}}) \times \dots \times (-\frac{1}{(i!)^{i!}}, \frac{1}{(i!)^{i!}}) \times \dots \in \mathcal{T}^{box}_{\prod\limits_{i=1}^{\infty}\mathbb{R}}$$

This neighborhood does not contain *any* element of the sequence. Indeed, for any n there exists i satisfying: $\frac{1}{(i!)^{i!}} < \frac{1}{(i\cdot n)^{i\cdot n}}$.

(b) We prove that X is path-connected (and thus connected). For this we construct the path from any point of X to the origin $\{0_\alpha\}_\alpha \in X$.
Fix some point $x \in X$, it has infinity of rational coordinates. Then we can split $A = A_1 \coprod A_2$ and present the point in the form $(\{x_\alpha\}_{\alpha\in A_1}, \{x_\beta\}_{\beta\in A_2}) \in X$ such that:
- both sets A_1, A_2 are infinite
- both $\{x_\alpha\}_{\alpha\in A_1}$ and $\{x_\beta\}_{\beta\in A_2}$ have infinity of rational coordinates.

Now consider the two paths:
- $\Big\{(\{x_\alpha\}_{\alpha\in A_1}, \{t\cdot x_\beta\}_{\beta\in A_2}) |\ t \in [0,1]\Big\}$. This path connects the point $(\{x_\alpha\}_{\alpha\in A_1}, \{0_\beta\}_{\beta\in A_2})$ to the point $(\{x_\alpha\}_{\alpha\in A_1}, \{x_\beta\}_{\beta\in A_2})$. By construction this path lies inside X.
- $\Big\{(\{t\cdot x_\alpha\}_{\alpha\in A_1}, \{0_\beta\}_{\beta\in A_2}) |\ t \in [0,1]\Big\}$. It connects the point $(\{x_\alpha\}_{\alpha\in A_1}, \{0_\beta\}_{\beta\in A_2})$ to the point $(\{0_\alpha\}_{\alpha\in A_1}, \{0_\beta\}_{\beta\in A_2})$ and again lies inside X.

The combination of the two paths connects the initial point to the origin.
It remains to check that both paths are continuous, i.e. that in both cases one speaks about a continuous function $[0,1] \xrightarrow{f} X$. As the topology is the product topology, the continuity is ensured by the continuity of each coordinate.

(3) (a) Take $A = (-1,1) \times \{0,..,0\} \subset \mathbb{R}^n$ with the open neighborhood $\mathcal{U} = Ball_1(0,..,0)$. Then $\mathcal{U}$ does not contain any ϵ-neighborhood, it does not even contain the closure $\overline{A}$.

(b) Recall that the uniform topology is metrizable. Therefore the compactness for $\mathcal{T}^{uniform}_{\prod\limits_{i=1}^{\infty}[0,1]}$ coincides with the sequential compactness. And the local compactness coincides with the local sequential compactness.
We show that $\mathcal{T}^{uniform}_{\prod\limits_{i=1}^{\infty}[0,1]}$, with the metric $\rho(\{x_i\},\{y_i\}) = sup_i\ d(\{x_i\},\{y_i\})$, is not locally sequentially compact at *any* point.

First we check the origin. Take any open neighborhood of the origin, $\mathcal{U}$, then $\mathcal{U}$ contains some open ball of radius $2\epsilon > 0$. Consider the sequence of points $\left\{pt^{(n)} = (\underbrace{0,\dots,0}_{n},\epsilon,0,0\dots)\right\}_n$. This sequence lies inside the ball (and hence in $\mathcal{U}$) and has no convergent subsequence because $d(pt^{(n)},pt^{(m)}) = \epsilon$ for any m,n. Thus $\mathcal{T}^{uniform}_{\prod\limits_{i=1}^{\infty}[0,1]}$ is not locally compact at the origin.

For the other points the construction is similar.