INTRODUCTION TO TOPOLOGY, SKETCHY SOLUTIONS OF MOED.B, (29.07.2016)

(1) (a) Consider the projection $X\times Y\stackrel{\pi_X}{\to}X$. Its restriction $\Gamma_f\stackrel{\pi_X}{\to}X$ is a bijection of sets and is continuous. The inverse of π_X is $X\stackrel{(Id,f)}{\to}\Gamma_f$, also continuous, thus π_X is a homeomorphism.

(b) Examples with f discontinuous but Γ_f connected:

- $[0,1]\stackrel{f}{\to}\mathbb{R}$, $f(x)=\begin{cases} sin\frac{1}{x}, & x\neq 0\\ 1, & x=0\end{cases}$.
- $\mathbb{R}^2\setminus(0,0)\stackrel{f}{\to}[0,2\pi]$, $f(x,y)=\phi$ (the angle in the polar coordinates).
- $S^1\to[0,2\pi)$, $z\to\frac{1}{2\pi i}ln(z)$.

(2) (a) f is continuous at $x\in X$ iff $f_A^{-1}(f_A(x))$ contains an open neighborhood of x. Therefore f is continuous at $x\in A$ iff $x\in Int(A)$. Similarly, f_A is continuous at $x\in X\setminus A$ iff $x\in Int(X\setminus A)$. Therefore the set of discontinuity points of f_A is: $\overline{A}\setminus Int(A)=\partial(A)$.

(b) The space $(0,1)$ (with the ordinary topology) satisfies the conditions, but is non-complete.

(3) (a) First note that f is continuous.

Solution 1.

Consider the function $g(x)=d(x,f(x))$. This function is continuous (the metric is continuous, as was shown in the class), thus achieves its infimum on the compact space X at some point, say x_0. If $inf_{x\in X}(g(x))=0$ then $d(x_0,f(x_0))=0$, i.e. $f(x_0)=x_0$. If $inf_{x\in X}(g(x))>0$ then $d(x_0,f(x_0))>d(f(x_0),f(f(x_0)))$, which is a contradiction.

Solution 2.

Denote $A=\bigcap\limits_{n=1}^{\infty}f^{(n)}(X)$, this is non-empty being the intersection of closed sets. We claim: $f(A)=A$. The part $f(A)\subseteq A$ is obvious. For the other part fix any $a\in A$, so that for any n holds: $a=f^{(n)}(x_n)$, for some $x_n\in X$. But then $a=f(f^{(n-1)}(x_n))$, i.e. $a\in f(\cap f^{n-1}(X))=f(A)$.

Now we claim: $diam(A)=0$. Suppose $diam(A)>0$. Note that $A\subset X$ is compact, being the intersection of compact subsets. Then exist $a_1,a_2\in A$ such that $d(a_1,a_2)=diam(A)$. But then $d(f(a_1),f(a_2))<d(a_1,a_2)$. As this hods for any pairs of points realizing the distance $diam(A)$, we get: $diam(f(A))<diam(A)$, contradicting $f(A)=A$.

Thus $diam(A)=0$, hence A has precisely one point, $A=\{a\}$. But then $f(a)=.a$

(b) *Solution 1.* Suppose the south pole of the sphere is not covered by γ. Fix the polar coordinates, $\phi\in[0,\pi]$, $\theta\in[0,2\pi)$ and define the homotopy $(\phi_t,\theta_t)=((1-t)\cdot\phi,\theta)$. It contracts $S^2\setminus\{\text{south pole}\}$ to one point (the north pole). In particular it contracts the loop γ.

Solution 2. Note that $S^2\setminus\{pt\}\approx\mathbb{R}^2$. Therefore $\pi_1(S^2\setminus\{pt\})=\pi_1(\mathbb{R}^2)=\{1\}$. Thus every loop in $S^2\setminus\{pt\}$ is contractible.

(4) (a) $\Rrightarrow$ Fix some open subset $\mathcal{U}\subseteq[0,1]$ and suppose at most a finite number of $\{\gamma_\alpha\}$ is non-constant on $\mathcal{U}$. Then for any $t\in\mathcal{U}$ and for any basic open neighborhood $\prod\limits_{\alpha\in A}\mathcal{V}_\alpha$ of $\{\gamma_\alpha(t)\}\in\prod\limits_{\alpha\in A}\mathbb{R}$ the preimage, $\{\gamma_\alpha\}^{-1}(\prod\limits_{\alpha\in A}\mathcal{V}_\alpha)=\bigcap\limits_{\alpha\in A}\gamma_\alpha^{-1}(\mathcal{V}_\alpha)$, is a *finite* intersection of open subsets of $[0,1]$, hence is open. Thus $\{\gamma_\alpha\}$ is continuous at t (in the box topology).

$\Lleftarrow$ Suppose the condition on $\{\gamma_\alpha\}$ does not hold, i.e. for some point $t_0\in[0,1]$ and some converging sequence $t_0\leftarrow t_k\in[0,1]$ there exists a sequence of indices, $\{\alpha_k\}$ for which holds: $\gamma_{\alpha_k}(t_0)\neq\gamma_{\alpha_k}(t_k)$. Take the neighborhoods $t_0\in\mathcal{U}_{\alpha_k}\not\ni t_{\alpha_k}$ and the basic open set $\Big(\prod\limits_k\gamma_{\alpha_k}(\mathcal{U}_{\alpha_k})\Big)\times\Big(\prod\limits_{\alpha\in A\setminus\{\alpha_k\}}\mathbb{R}\Big)$. Then the preimage $\{\gamma_\alpha\}^{-1}(\Big(\prod\limits_k\gamma_{\alpha_k}(\mathcal{U}_{\alpha_k})\Big)\times\Big(\prod\limits_{\alpha\in A\setminus\{\alpha_k\}}\mathbb{R}\Big))\subset[0,1]$ contains t_0 but does not contain any of the points t_k. Thus t_0 does not lie in the interior of $\{\gamma_\alpha\}^{-1}(\Big(\prod\limits_k\gamma_{\alpha_k}(\mathcal{U}_{\alpha_k})\Big)\times\Big(\prod\limits_{\alpha\in A\setminus\{\alpha_k\}}\mathbb{R}\Big))\subset[0,1]$, i.e. this preimage does not contain any open neighborhood of t_0. But then the map $\{\gamma_\alpha\}$ is not continuous at t_0.

(b) It is enough to prove that all the basic open sets are not path-connected. Fix some basic open $\prod_{\alpha\in A}\mathcal{U}_\alpha$, and choose some points $\{x_\alpha\}$, $\{y_\alpha\}$ with the condition: $x_\alpha \neq y_\alpha$ for infinity of values of α. We prove that there does not exists a path from $\{x_\alpha\}$ to $\{y_\alpha\}$.
Indeed, any such path is a continuous map $\{\gamma_\alpha\}$ satisfying: $\{\gamma_\alpha(0)\} = \{x_\alpha\}$ and $\{\gamma_\alpha(1)\} = \{y_\alpha\}$. By the part 4.a, for any t_0 there exists a neighborhood $t_0 \in \mathcal{V}_{t_0} \subseteq [0,1]$ on which at most a finite number of γ_α are non-constant. We cover $[0,1]$ by such neighborhoods. Using the compactness of $[0,1]$ we choose a finite subcover, $[0,1] = \cup_i \mathcal{V}_i$.
But then for the points $\{x_\alpha\}$, $\{y_\alpha\}$ at most finite number of coordinates can differ. This brings the contradiction.