מבוא לטופולוגיה, מבחן סופי (מועד א).
אוניברסיטת בן גוריון

מספר הקורס: 201.1.0091	כללים: אסור לכתוב בצבע אדום.
מרצה ד.קרנר	הבודק רוצה לראות רק את הגרסה הסופית של הפתרון,
תאריך: 08.07.2016	לא את כל נדודי הביניים. השתמשו בטיוטה לכל הנסיונות
משך הבחינה: 3 שעות	ההתחלתיים. הפתרון אמור להיות מסודר, מדויק (ולא ארוך).
ניקוד: פתרו את כל השאלות	בזמן הבחינה מרצים/מתרגלים עונים רק על שאלות הקשורות
(סה"כ 105 נקודות)	לניסוח של הבחינה. אנחנו לא עונים על שאלות כמו: "האם זאת
	דרך נכונה?" , "באיזה משפט צריכים להשתמש כאן?",
אין להשתמש בכל חומר עזר, לרבות מחשבונים	"אני שכחתי את הנוסחה/הניסוח של..".

בכל השאלות: תשובה "כן" דורשת הוכחה/בניה מפורשת, תשובה "לא"- הסבר מפורש/דוגמא נגדית

(1) (א) (10) נגדיר קבוצה $X = \left\{e^{\phi}(1-\frac{1}{100}) < r < e^{\phi}(1+\frac{1}{100}),\ \phi\in(-\infty,\infty)\right\} \subset \mathbb{R}^2$, כאן (r,ϕ) קואורדינטות קוטביות. האם X (עם הטופולוגיה המושרית) הומאומורפי ל $\mathbb{R}^2$?

(ב) (20) יהי (X,d) מרחב קשיר מסילתית (עם לפחות שתי נקודות). הוכיחו ש X הנו גדול מבן מניה בעוצמתו.

(2) (10) תהי $X = C[0,1]$ קבוצת פונקיות רציפות ב $[0,1]$ ונגדיר $d(f,g) = \underset{x\in[0,1]}{sup}|f(x)-g(x)|$.
האם (X,d) הנו מרחב שלם?

(3) (א) (20) נתונות העתקות רציפות, $\left\{S^1 \overset{f_\alpha}{\to} \prod_{\beta\in B} S^n\right\}_{\alpha\in A}$, כאשר $n\in\mathbb{N}$ קבוע, A,B קבוצות אינסופיות ו $\prod_{\beta\in B} S^n$ מרחב עם טופולוגית מכפלה. נניח שעבור כל תת-קבוצה סופית $I\subset A$ מתקיים: $\underset{\alpha\in I}{\cap} f_\alpha(S^1)\neq\varnothing$.
הוכיחו: $\underset{\alpha\in A}{\cap} f_\alpha(S^1)\neq\varnothing$.

(ב) (20) יהי $X=(0,1]$ עם טופולוגיה רגילה. האם קיימת קומפקטיפיקצית Hausdorff $\bar{X}$ של X שעבורה $\bar{X}\setminus X$ הומאומורפי ל $[0,1]^2$?

(4) (א) (5) חשבו $\pi_1\Big(\mathbb{R}^3\setminus(0,0,0)\Big)$.
(ב) (20) חשבו $\pi_1\Big(\mathbb{R}^3\setminus\big\{(0,0,1),(0,0,-1)\big\}\Big)$.

בהצלחה!

INTRODUCTION TO TOPOLOGY,
SKETCHY SOLUTIONS OF MOED.A, (08.07.2016)

(1) (a) Draw $X \subset \mathbb{R}^2$, in the x, y coordinates this is a neighborhood of the spiral $\{r = e^{\phi},\ \phi \in (-\infty, \infty)\}$. In the r, ϕ coordinates this is an infinite strip whose width tends to zero as $\phi \to -\infty$ and tends to infinity as $\phi \to \infty$.
Define the map $X \stackrel{f}{\to} (-1, 1) \times \mathbb{R}$ by $f(r, \phi) = (100\frac{r-e^{\phi}}{e^{\phi}}, \phi)$. This map is continuous, injective and (by direct check) surjective. Its inverse is continuous as well, thus f is a homeomorphism. Finally, using $(0, 1) \stackrel{homeo}{\approx} \mathbb{R}$ we get: $X \stackrel{homeo}{\approx} \mathbb{R}^2$.

(b) Suppose (X, d) is countable, we show that X is non-connected. Fix some point $x_0 \in X$ and a number $r < \frac{diam(X)}{2}$ such that for any other point $x \in X$: $d(x_0, x) \neq r$. (Such r exists because X is countable.) Then the open ball $Ball_r(x_0)$ does not contain the whole X and no points of X belong to the boundary of $Ball_r(x_0)$. Further, the subset $U_{>r} = \{x |\ d(x, x_0) > r\}$ is non-empty and open. Thus we get a separation into the disjoint open subsets $X = Ball_r(x_0) \coprod U_{>r}$. This means X is non-connected, giving a contradiction.
<u>Another solution.</u> Fix any two points $x_1, x_2 \in X$ and take a path γ from x_1, x_2. This is a compact subset of a metric space, in particular Hausdorff. As has been proved in a lecture: "a compact Hausdorff space with no isolated points is uncountable".
<u>Another solution.</u> Fix any two points $x_1, x_2 \in X$ and take a path γ from x_1, x_2. Consider the function $\gamma \stackrel{dist}{\to} \mathbb{R}$, defined by $dist(x) = d(x_1, x)$. This function is continuous and $dist(x_1) = 0 \neq dist(x_2)$. Its image contains the interval $[0, dist(x_2)] \subset \mathbb{R}$. (Suppose $[0, dist(x_2)] \ni c \notin dist(\gamma)$, then $dist^{-1}[0, c) \coprod dist^{-1}(c, dist(x_2)]$ is a separation of γ into disjoint open subsets.) Thus γ must have at least a continuum of points.

(2) Let $\left\{[0, 1] \stackrel{f_n}{\to} \mathbb{R}\right\}_n$ be a Cauchy sequence for the metric d. For each $x \in X$ the sequence of points $\{f_n(x) \in \mathbb{R}\}$ converges, by the completeness of $\mathbb{R}$. Define the function f pointwise, $f(x) = \lim f_n(x)$.
We claim that the convergence $f_n \to f$ is not just pointwise, but uniform. This follows immediately by the form of the metric. Finally, as the convergence is the uniform, the limit f is a continuous function (as has been proved in Infi). Therefore (X, d) is complete.

(3) (a) By Tychonoff's theorem the space $\prod\limits_{\beta \in B} S^n$ is compact in the product topology. For each α the image $f_\alpha(S^1) \subset \prod\limits_{\beta \in B} S^n$ is a compact subset (being a continuous image of a compact space). Thus $f_\alpha(S^1) \subset \prod\limits_{\beta \in B} S^n$ is a closed subset. Therefore we have a collection of closed subsets, $\{f_\alpha(S^1) \subset \prod\limits_{\beta \in B} S^n\}_{\alpha \in A}$, with the property of non-empty finite intersections. Thus, by compactness of $\prod\limits_{\beta \in B} S^n$, the total intersection is non-empty.

(b) We construct the needed compactification as the closure, $\overline{\Gamma_f}$, of the graph of a map $X \stackrel{f}{\to} [0, 1]^2$.
Present $(0, 1] = \bigcup\limits_{n=1}^{\infty} [\frac{1}{n+1}, \frac{1}{n}]$ and define f on each $[\frac{1}{n+1}, \frac{1}{n}]$ as the Peano curve. More precisely: $f|_{[\frac{1}{n+1}, \frac{1}{n}]}$ is the Peano curve that begins at the point $(n\ mod(2), 0) \in [0, 1]^2$ and ends at the point $(n + 1\ mod(2), 0) \in [0, 1]^2$. By the construction these maps glue to a continuous map $X \stackrel{f}{\to} [0, 1]^2$. This map is surjective on each segment $[\frac{1}{n+1}, \frac{1}{n}]$, therefore the set of partial limits of $f(t)$ as $t \to 0$ is precisely $[0, 1]^2$.
Finally, $\bar{X} \subset \mathbb{R}^3$, in particular it is Hausdorff.
<u>Another version of f.</u> Define the map $(0, 1] \stackrel{f}{\to} [0, 1]^2$ by $f(x) = (|sin\frac{1}{x}|, |sin\sqrt{\frac{1}{x}}|)$. Check this map on the intervals $[\frac{1}{t^2}, \frac{1}{(t+\delta)^2}]$, as $t \to \infty$, while $0 < \delta$ is small, but fixed. On each such interval the y_2 coordinate changes slightly, while the y_1 coordinate oscillates.

(4) We claim that in both cases all the loops are contractible and therefore $\pi_1(..) = \{1\}$.
As $\mathbb{R}^3 \setminus (...)$ is connected, it is enough to consider the loops based at e.g. $(1, 0, 0)$.

(a) Apply the homotopy $\big(\mathbb{R}^3 \setminus (0,0,0)\big) \times [0,1] \stackrel{F(x,t)}{\to} \mathbb{R}^3 \setminus (0,0,0)$ defined in polar coordinates by

$$\phi(t) = \phi, \quad \theta(t) = \theta, \quad r(t) = r(1-t) + t.$$

This homotopy pushes every loop in $\mathbb{R}^3 \setminus (0,0,0)$ to a loop on S^2. (Note that the base point, $(1,0,0) \in S^2$, remains fixed.) And any loop on the sphere is contractible, as has been shown in the class/homeworks.

(b) Fix some loop $\gamma \subset \mathbb{R}^3 \setminus \big\{(0,0,1),(0,0,-1)\big\}$, defined by $[0,1] \stackrel{f}{\to} \mathbb{R}^3 \setminus \big\{(0,0,1),(0,0,-1)\big\}$, $f(0) = f(1) = (1,0,0)$.
First we construct the homotopy that pushes-out γ to the cylinder $\{x^2 + y^2 = 1\} \subset \mathbb{R}^3$.

- If γ does not cross the $\hat{z}$-axis then the homotopy is induced by the homotopy $\mathbb{R}^2 \setminus (0,0) \rightsquigarrow S^1$, $(r, \phi) \to (r(1-t) + t, \phi)$.
- In the general case one considers a small punctured cylinder $D^2_\epsilon \times (-\infty, 0] \setminus (0,0,-1)$ and its preimage $f^{-1}(D^2_\epsilon \times (-\infty, 0] \setminus (0,0,-1))$. The later is an open subset of $[0,1]$, hence splits into the union $\cup(a_i, b_i)$. On each (a_i, b_i) one deforms f slightly, so that the deformed path does not cross the $\hat{z}$-axis. Do the same for the cylinder $D^2_\epsilon \times [0,\infty) \setminus (0,0,1)$ Then apply the homotopy as above.

Now we get a loop on the cylinder $\partial(D^2_\epsilon) \times \mathbb{R}$, which can be shrank into a loop inside the circle $\{z = 0,\ x^2 + y^2 = 1\} \subset \mathbb{R}^3$.
Finally, we contract this circle to the point $(1,0,0)$.

Another version of homotopy. As γ does not pass through the points $(0,0,1), (0,0,-1)$ it has a positive distance from both of them. Thus take small spheres centered at each of these points and inflate them up to radius 1. This pushes any loop inside $\mathbb{R}^3 \setminus \big\{(0,0,1),(0,0,-1)\big\}$ to a loop inside

$$\mathbb{R}^3 \setminus \big\{Ball_1(0,0,1) \cup Ball_1(0,0,-1)\big\}.$$

Now, if the loop passes through the point $(0,0,0)$, it can be moved off this point. After this the loop can be pushed off the ball $Ball_1(0,0,0)$. Thus we get a loop inside $\mathbb{R}^3 \setminus Ball_1(0,0,0)$ and this is contractible as in part a.