Introduction to Topology, 201.1.0091

Homework 10

Spring 2016 (D.Kerner)

(1) Prove: (X, d) is complete iff for any decreasing sequence of closed sets, $A_1 \supset A_2 \supset \cdots$, with $diam(A_n) \to 0$, holds: $\bigcap_{n=1}^{\infty} A_n$ is one point.

(2) (a) Given a topological space $(X, \mathcal{T}_X)$ and a complete metric space (Y, d_Y), consider the space of all the functions from X to Y, $Func(X, Y)$, with the uniform metric $\bar{\rho}(f, g) = sup_{x \in X} \bar{d}_Y(f(x), g(x))$. (Check that this is a metric.) Check that the convergence in this metric is precisely the uniform convergence of functions, $\{f_n\} \to f(x)$.

(b) Prove that the subspaces of continuous functions, $C(X, Y)$, bounded functions, $\mathcal{B}(X, Y)$, are complete.

(c) Denote by $V_r \subset C((0, 1), \mathbb{R})$ the subspace of r-times differentiable functions. Is it complete?

(d) For real valued sequences define $d_p(\{x_i\}, \{y_i\}) = \sqrt[p]{\sum |x_i - y_i|^p}$ and $d_\infty(\{x_i\}, \{y_i\}) = sup\{|x_i - y_i|\}$. Prove that the metric spaces $(l_p, d_p) = \Big\{\{x_n\} | \ d_p(\{x_n\}, 0) < \infty\Big\}$ are complete for $1 \leq p \leq \infty$. What about the spaces $(l_0, d_\infty) = \Big\{\{x_n\} | \ x_n \to 0\Big\}$, $\quad (l_{00}, d_p) = \Big\{\{x_n\} | \ x_n = 0 \ for \ n \gg 0\Big\}$?

(3) Let $C[0, 1]$ be the space of continuous functions on $[0, 1]$. Prove that $||f||_p := (\int_0^1 |f(x)|^p dx)^{\frac{1}{p}}$, for $p \geq 1$, defines a norm on $C[0, 1]$, and then a metric. (You can use Minkowski's inequality.) Is $C[0, 1]$ complete in this metric?

(4) The uniform continuity of a function $(X, d_X) \xrightarrow{f} (Y, d_Y)$ is defined as in Calculus.

(a) Prove: if (X, d) is compact then every continuous f is uniformly continuous.

(b) Let $A \subset X$ and suppose Y is complete. Show that if $A \xrightarrow{f} Y$ is uniformly continuous then it extends (uniquely) to a continuous function $\bar{A} \xrightarrow{\bar{f}} Y$ and $\bar{f}$ is also uniformly continuous.

(5) Two metrics d, d' on X are said to be metrically equivalent if both the identity map $(X, d) \xrightarrow{Id} (X, d')$ and its inverse are uniformly continuous.

(a) Show that d is metrically equivalent to $\bar{d} = min(d, 1)$.

(b) Suppose d, d' are metrically equivalent. Show that (X, d) is complete iff (X, d') is complete. Does this hold if Id and its inverse are just continuous (i.e. with no uniform continuity assumption)?

(c) Give examples of metrically non-equivalent metrics on $\mathbb{R}$, $(0, 1)$.

(6) (Banach's fixed point theorem)

(a) A map $(X, d) \xrightarrow{f} (X, d)$ is called a *contracting* if for some fixed $\alpha < 1$ holds: $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$ for any $x, y \in X$. Prove that any contracting map of a complete metric space always has a fixed point (i.e. $f(x) = x$).

(b) What happens if X is not complete? (Construct contractions without fixed points for $X = (0, \infty)$, $X = \mathbb{Q}$,...)

(c) Does the statement hold with the contracting condition weakened to: "$d(f(x), f(y)) < d(x, y)$ for any $x, y \in X$"?

(d) Prove: if X is compact then the condition "$d(f(x), f(y)) < d(x, y)$ for any $x, y \in X$" ensures the unique fixed point.

(e) Suppose $f^{(k)} = \underbrace{f \circ f \circ \cdot \circ f}_{k}$ is a contraction (for some k) and X is complete. Prove that for any x the sequence $f^{(n)}(x)$ converges to the unique fixed point of f. (e.g. $cos(x)$ is not a contraction, but $cos^{(2)}(x)$ is a contration.)

(7) Given a sequence of metric spaces, $\{(X_n, d_n)\}_n$, take the product, $\prod X_n$, with the metric $d(\{x_n\}, \{y_n\}) = sup_n\{\frac{\bar{d}_n(x_n, y_n)}{n}\}$. Suppose each (X_n, d_n) is totally bounded. Prove that the product is totally bounded. Conclude (without using Tychonoff's theorem) that a countable product of compact metrizable spaces is compact.

(8) (a) For any n construct a continuous surjective map $[0, 1] \xrightarrow{f} [0, 1]^n$. (This can be done in various ways, in particular do the "direct" construction, $[0, 1] \twoheadrightarrow [0, 1]^n$, and the "inductive" one, $[0, 1] \twoheadrightarrow [0, 1]^2 \twoheadrightarrow [0, 1]^2 \times [0, 1] \twoheadrightarrow \cdots$)

(b) For any n construct a continuous surjective map $\mathbb{R} \xrightarrow{f} \mathbb{R}^n$.

(9) (a) (another construction of completion) Show that any space (X, d) can be embedded isometrically into a complete space (Y, d) as follows. Define the equvivalence on X by $\{x_n\} \sim \{\tilde{x}_n\}$ if $d(x_n, \tilde{x}_n) \to 0$. Denote by Y the set of all the equivalence classes of Cauchy sequences, with the metric $d\Big([\{x_n\}], [\{\tilde{x}_n\}]\Big) = \lim_{n \to \infty} d(x_n, \tilde{x}_n)$.

(i) Show that $\sim$ is an equivalence relation and $d([..], [..])$ is a well defined metric.

(ii) Define $X \xrightarrow{h} Y$ by $x \to [(x, x, \dots)]$. Prove that h is an isometric embedding.

(iii) Show that $h(X)$ is dense in Y.

(iv) Show that if $A \subset Z$ is a dense subset of a metric space and if every Cauchy sequence of A converges (in Z), then Z is complete. Achieve from here that Y (as defined above) is complete.

(b) (Uniqueness of completion) Given two isometric embeddings $\tilde{Y} \overset{\tilde{h}}{\hookleftarrow} X \overset{h}{\hookrightarrow} Y$ into complete metric spaces $Y, \tilde{Y}$, show that there exists an isometry $\overline{h(X)} \to \overline{\tilde{h}(X)}$ whose restriction onto $h(X)$ coincides with $\tilde{h} \circ h^{-1}$.