Introduction to Topology, 201.1.0091

Homework 11

Spring 2016 (D.Kerner)

(1) (a) Given a surjective map, $(X,\mathcal{T}_X)\stackrel{p}{\twoheadrightarrow}Y$, define $\mathcal{T}_Y=\{\mathcal{U}|\ p^{-1}(\mathcal{U})\in\mathcal{T}_X\}$. Check that $\mathcal{T}_Y$ is a topology. Prove that $\mathcal{T}_Y$ is the coarsest topology for which p is a quotient map.

(b) Given a homeomorphism $X\stackrel{f}{\xrightarrow{\sim}}Y$ and an equivalence relation $\sim$ on X. Prove the homeomorphism $X/_\sim\approx Y/_{f(\sim)}$.

(c) Let Y be a (topological) quotient of X. Prove/disprove: if X is Hausdorff/connected/path-connected/compact then Y is Hausdorff/connected/path-connected/compact.

(2) (a) Construct the quotient map from $[0,1]$ onto every (capital) letter of Latin alphabet.

(b) Prove that every finite connected graph can be obtained as a quotient space of $[0,1]$.

(c) Construct the homeomorphism $D^n/_{S^{n-1}}\xrightarrow{\sim}S^n$ (here D^n is the closed disc whose boundary is S^{n-1}).

(d) Prove that the following spaces are homeomorhic (here $X/_A$ denotes the contraction of A to a point): $\mathbb{R}^2$, $\mathbb{R}^2/_{Ball_r(0,0)}$, $\mathbb{R}^2/_{[0,1]\times[0,1]}$, $\mathbb{R}^2/_{[0,1]\times\{0\}}$, $\mathbb{R}^2/_{star}$ (here $star\subset\mathbb{R}^2$ is a wedge of a finite number of straight segments), $\mathbb{R}^2/_{chain}$ (here $chain\subset\mathbb{R}^2$ is a finite bounded piecewise linear path with no self-crossings and repetitions).

(e) Are the spaces $\mathbb{R}^2$, $\mathbb{R}^2/_{ray}$ homeomorphic? (Here $ray\subset\mathbb{R}^2$ is e.g. $\{y=0,\ x\geq 0\}$).

(f) Describe the quotients of $\mathbb{R}^2$ by the following equivalences: i. $(x_1,y_1)\sim(x_2,y_2)$ if $x_1=x_2$. ii. $(x_1,y_1)\sim(x_2,y_2)$ if $x_1+y_1^2=x_2+y_2^2$. iii. $(x_1,y_1)\sim(x_2,y_2)$ if $x_1^2+y_1^2=x_2^2+y_2^2$.

(g) Fix n non-intersecting circles on S^2. Identify the space obtained by contracting these circles to one point.

(3) (a) Identify the following quotients (write down the explicit subdivision of $[0,1]^2$ in each case)

i. $[0,1]^2\Big/\Big((0,t)\sim(1,t),\ \forall\ t\in 0,1]\Big)$, ii. $[0,1]^2\Big/\Big((0,t)\sim(1,1-t),\ \forall\ t\in 0,1]\Big)$, iii. $[0,1]^2\Big/\Big({(0,t)\sim(1,t), \atop (t,0)\sim(t,1)}\ \forall\ t\in 0,1]\Big)$.

(b) Idenitfy the surface $S_n\subset\mathbb{R}^3$ defined as the image of $[0,2\pi]\times[-1,1]$ by $(x(\phi,t),y(\phi,t),z(\phi,t))=\Big((R+t\cdot cos(\frac{n\phi}{2}))cos(\phi),(R+t\cdot cos(\frac{n\phi}{2}))sin(\phi),t\cdot sin(\frac{n\phi}{2})\Big)$. (Here $R>1$ and $n\in\mathbb{Z}_{>0}$ are fixed.) Prove that all $\{S_n\}_{n\in 2\mathbb{N}}$ are homeomorphic and all $\{S_n\}_{n\in 2\mathbb{N}+1}$ are homeomorphic. (Here the homeomorphisms are of surfaces only, not of their embeddings.)

(4) (a) Given X with two equivalences, $\stackrel{1}{\sim}$, $\stackrel{2}{\sim}$, prove: $X/_{(\stackrel{1}{\sim},\stackrel{2}{\sim})}\xrightarrow{\sim}X/_{\stackrel{1}{\sim}}\Big/_{\stackrel{2}{\sim}/_{\stackrel{1}{\sim}}}$. (Here the equivalences $(\stackrel{1}{\sim},\stackrel{2}{\sim})$, $\stackrel{2}{\sim}/_{\stackrel{1}{\sim}}$ were defined in the class.)

(b) Prove that the Klein bottle can be obtained (in a double quotient way) by either gluing the boundary of Moebius strip or gluing the (twisted) boundary of a cylinder.

(5) Prove that the spaces $D^2/\Big((x,y)\sim(-x,y)\Big)$, $D^2/\Big((x,y)\sim(-x,-y)\Big)$ are homeomorphic to D^2. What is the generalization to D^n?

(6) (a) Prove that the following is an equivalent definition of quotient map: $X\stackrel{p}{\twoheadrightarrow}Y$ is a quotient map if: $V\subset Y$ is closed iff $p^{-1}(V)\subset X$ is closed.

(b) Recall: a continuous map $X\stackrel{f}{\to}Y$ is open (closed) if $f(open)=open$ (or $f(closed)=closed$). Is the projection onto x-axis, $\mathbb{R}^2\stackrel{\pi_x}{\to}\mathbb{R}$, a closed map? Is the embedding of x-axis, $\mathbb{R}\stackrel{i}{\hookrightarrow}\mathbb{R}^2$, an open map?

(c) Prove that an open/closed map is always a quotient map. Give an example of quotient map which is neither closed nor open.

(d) Given a quotient map, $X\stackrel{p}{\to}Y$, and a saturated subset $A\subset X$ (i.e. $p^{-1}p(A)=A$). Suppose: either A is open/closed or p is open/closed. Prove that $A\stackrel{p|_A}{\to}p(A)$ is a quotient map. (Without the assumptions on p or A the map $p|_A$ is not necessarily quotient, see Munkres §22.)

(e) Given a quotient map $X\stackrel{p}{\to}Y$ and some other map $X\stackrel{f}{\to}Z$ prove: "f descends to Y" (i.e. exists $Y\stackrel{p_*f}{\to}Z$ satisfying: $(p_*f)\circ p=f$) iff f is constant on the fibres of p, i.e. if $p(x_1)=p(x_2)$ then $f(x_1)=f(x_2)$.

(f) Construct the quotient map from $[0,1]$ onto $[0,1]^2$. (Hint: prove that Peano curve is defined by closed map.)

(7) Let G be a topological group (a group which is a T_1-topological space and such that the product and the inverse operations are continuous).

(a) Check that the group $(\mathbb{C},+)$ is a topological group (for the standard topology) and $(\mathbb{Z},+)\subset(\mathbb{Q},+)\subset(\mathbb{R},+)\subset(\mathbb{C},+)$ are topological subgroups (for the induced topology).

(b) Let $\Bbbk=\mathbb{R},\mathbb{C}$ with the standard topology. The isomorphism of vector spaces $Mat_{n\times n}(\Bbbk)\approx\Bbbk^{n^2}$ defines the topology on $Mat_{n\times n}(\Bbbk)$. Check that the following groups are topological (with the topology induced from $Mat_{n\times n}(\Bbbk)$): $GL_n(\Bbbk)$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$. Are they connected/compact?

(c) Which of the following subgroups of $\mathbb{R}$ are closed subsets: $\mathbb{Z}$, $\sqrt{2}\cdot\mathbb{Z}$, $\mathbb{Z}+\sqrt{2}\cdot\mathbb{Z}$?

(d) Prove that for any $a\in G$ the maps $G\stackrel{\phi_a,\psi_a,\tau_a}{\to}G$, $\phi_a(x)=a^{-1}x$, $\psi_a(x)=xa$, $\tau_a(x)=a^{-1}xa$ are homeomorphisms. Moreover, τ_a is a group isomorphism.

(e) Prove that for any subgroup $H\subset G$ the closure $\overline{H}\subseteq G$ is also a topological subgroup.

(f) Fix a subgroup $H\subset G$ and consider the set of left cosets, $\{xH\}_{x\in G}$. Denote it by $G/_H$ and give it the quotient topology. Show that the quotient map $G\to G/_H$ is open. Show that the one-point sets of $G/_H$ are closed iff H is closed.

(g) Show that if $H\subset G$ is a normal subgroup then $G/_H$ is a topological group. What are $(\mathbb{R},+)/_{(\mathbb{Z},+)}$, $(\mathbb{R}^n,+)/_{(\mathbb{Z}^n,+)}$?