Introduction to Topology, 201.1.0091

Homework 12

Spring 2016 (D.Kerner)

(1) (a) Construct an explicit homotopy from a(ny) map $[0,1]\stackrel{f}{\to}[0,1]$ satisfying $f(0)=0$, $f(1)=1$ to the identity map, $Id(x)=x$.

(b) Construct an explicit homotopy between any two paths $[0,1]\stackrel{f,g}{\to}\mathbb{R}^2$ connecting $(-1,0)$ to $(1,0)$.

(c) Prove that path homotopy (in arbitrary topological space) is an equivalence relation.

(d) Prove that the product of homotopy types of paths, $[f]\cdot[g]=[f\star g]$, is well defined.

(e) Denote by e_{x_0}, e_{x_1} the constant paths at x_0,x_1. For a path f from x_0 to x_1 prove that $e_{x_0}\star f\sim f$ and $f\star e_{x_1}\sim f$. (Note that $e_{x_0}\star f\neq f$ and $f\star e_{x_1}\neq f$.)

(f) Given the paths $[0,1]\stackrel{f,g,h}{\to}X$ such that $f(1)=g(0)$ and $g(1)=h(0)$ prove: $[f\star(g\star h)]=[(f\star g)\star h]$. (Note that $f\star(g\star h)\neq(f\star g)\star h$.)

(g) Given a path f from x_0 to x_1 define $f^{-1}(s)=f(1-s)$. Prove: $f\star f^{-1}\sim e_{x_0}$ and $f^{-1}\star f\sim e_{x_1}$. (Do the equalities hold here?)

(h) Suppose the maps $X\stackrel{f,f'}{\to}Y$ are homotopic and the maps $Y\stackrel{g,g'}{\to}Z$ are homotopic. Construct an explicit homotopy of the maps $g\circ f$, $g'\circ f'$.

(2) (a) Prove that every loop in S^n that misses at least one point (i.e. is non-surjective), is contractible. (Construct an explicit homotopy to the constant loop.)

(b) Prove: for any map $[0,1]\stackrel{f}{\to}S^{n>1}$ there exists a subdivision $0=s_0<s_1<\cdots<s_N=1$ such that each path $[s_i,s_{i+1}]\stackrel{f}{\to}S^n$ is homotopic to an arc connecting the points $f(s_i),f(s_{i+1})$. Achieve from here: any map $[0,1]\stackrel{f}{\to}S^{n>1}$ is homotopic to a non-surjective map. Thus $\pi_1(S^{n>1})=\{1\}$.

(c) Prove that any two non-surjective maps $X\stackrel{f,f'}{\to}S^{n>0}$ are homotopic. What about $X\stackrel{f,f'}{\to}S^0$?

(3) A subset $A\subset\mathbb{R}^n$ is called "a star" if for some point $a_0\in A$ any segment from a_0 to any other point of A lies inside A.

(a) Prove that any two maps to a star, $X\stackrel{f,f'}{\to}A$, are homotopic. (In particular any star-set is simply connected.)

(b) Is it true that any two maps from a star set to a path-connected space, $A\stackrel{f,f'}{\to}X$ are homotopic?

(4) X is called contractible if the identity map $X\stackrel{Id}{\to}X$ is nullhomotopic (i.e. homotopic to a constant map, $X\to pt\in X$).

(a) Prove that the following spaces are contractible: $\mathbb{R}^n$, an open/closed ball in $\mathbb{R}^n$, $\{(x,y,z)|\ z^2=x^2+y^2\}\subset\mathbb{R}^3$.

(b) Prove that being contractible is a topological property (i.e. if $X\stackrel{homeo}{\approx}Y$ then X is contractible iff Y is contractible).

(c) Is a (finite/infinite) product of contractible spaces contractible? (in the product topology) Is any open/closed subspace of a contractible space contractible?

(5) Prove: the maps $X\stackrel{f,f'}{\to}Y_1\times Y_2$ are homotopic iff $p_1(f)\sim p_1(f')$ and $p_2(f)\sim p_2(f')$.

(6) (a) Show that any map $X\stackrel{\phi}{\to}Y$, $\phi(x_0)=y_0$, induces a homomorphism of groups $\pi_1(X,x_0)\stackrel{\phi_*}{\to}\pi_1(Y,y_0)$. Which map $\pi_1(S^1)\stackrel{\phi_*}{\to}\pi_1(S^1)$ is induced by $z\to z^n$?

(b) Suppose ϕ is surjective/injective/homeomorphism, is ϕ_* necessarily surjective/injective/isomorphism? Suppose $X\stackrel{\phi}{\to}X$ is homotopic to the identity map, $X\stackrel{Id}{\to}X$, what is the map ϕ_*?

(c) For the maps of connected spaces $X\stackrel{\phi}{\to}Y\stackrel{\psi}{\to}Z$ prove: $(\psi\circ\phi)_*=\psi_*\circ\phi_*$.

(7) (a) Show that a map $\mathbb{R}^2\supseteq A\stackrel{f}{\to}Y$ (for A connected) extends continuously to the whole $\mathbb{R}^2$ iff the induced homomorphism $\pi_1(A)\stackrel{f_*}{\to}\pi_1(Y)$ is trivial.

(b) Prove: for any two polynomials of the same degree, $p(x),q(x)\in\mathbb{C}[x]$, there exists $r>0$ such that for any $R>r$ the maps $\{z|\ |z|=R\}\stackrel{p,q}{\to}\mathbb{C}\setminus(0,0)$ are homotopic.

(8) Suppose X is path-connected, prove that the following conditions are equivalent:

i. X is simply connected. ii. Any map $S^1\stackrel{f}{\to}X$ is homotopic to a constant map. iii. Any map $S^1\stackrel{f}{\to}X$ can be extended to a (continuous) map $D^2\stackrel{f}{\to}X$. iv. Any two paths from x_0 to x_1 in X are homotopic.

(9) Let $X\stackrel{p}{\to}B$ be a covering, with X path connected and B simply connected. Prove that the covering is a homeomorpism. (In particular, the spaces $\mathbb{R}^n$, D^n have only trivial coverings.)

(10) (a) Fix the matrix $A=\begin{bmatrix}n_1 & m_1\\ n_2 & m_2\end{bmatrix}\in Mat_{2\times2}(\mathbb{Z})$ and consider the map $S^1\times S^1\stackrel{\phi_A}{\to}S^1\times S^1$, $\phi_A(z,w)=(z^{n_1}w^{m_1},z^{n_2}w^{m_2})$. Prove that ϕ_A is homeomorphism iff $A\in GL(2,\mathbb{Z})$, i.e. $det(A)=\pm1$.

(b) Fix the standard basis of $\mathbb{Z}^2$. Draw/imagine the cycles on the torus whose classes in $\pi_1(S^1\times S^1)$ are $(1,0)$, $(1,1)$.

(c) Compute the presentation matrix of the map $\pi_1(S^1\times S^1)\stackrel{(\phi_A)_*}{\to}\pi_1(S^1\times S^1)$.

(d) Prove that there exists a self-homeomorphism of the torus that sends a loop of the class $(3,4)$ to a loop of the class $(1,0)$. Could you visualize it?

(11) Let $A\in Mat_{3\times3}(\mathbb{R}_{\geq0})$ and suppose that either all the entries of A are positive or $det(A)\neq0$. Prove: A has a positive (real) eigenvalue. (Part of this was proved in the class.)