Introduction to Topology, 201.1.0091
Homework 2

Spring 2016 (D.Kerner)

(1) Given a topological space $(X, \mathcal{T}_X)$ and a set $Y = X \coprod \{y_0\}$. Check that the following collection defines a topology on Y: $\mathcal{T}_Y := \Big\{ \{y_0\} \cup U | \ \forall\ U \in \mathcal{T}_X \Big\} \cup \{\varnothing\}$.
Check that in the space $(Y, \mathcal{T}_Y)$ the point y_0 belongs to the closure of every other point.

(2) (a) Give examples of topological spaces for which every open set is closed. (Prove that the later condition is equivalent to: 'every closed set is open'.) Prove that if such a space X is a topological subspace of $\mathbb{R}^n$ then the induced topology on X is discrete.
(b) Given an increasing sequence of real numbers, $a_1 < a_2 < \cdots$, prove that the subspace (with the induced topology) $X = \{\underset{i}{\cup}(a_{2i-1}, a_{2i})\} \subset \mathbb{R}$ has infinitely many subsets which are both open and closed.

(3) Given two topological spaces, X, Y, with closed subsets, $A \subset X$, $B \subset Y$, prove that $A \times B \subset X \times Y$ is a closed subset.

(4) Fix a sequence $A := \{a_1, a_2, \dots\} \subset \mathbb{R}$. Denote by $Part.Lim(A)$ the set of all the partial limits of $\{a_n\}$.
(a) Give examples of sequences for which:
i. $Part.Lim(A) = A$. ii. $Part.Lim(A) \cap A = \varnothing$. iii. $Part.Lim(A) = \mathbb{R}$.
(b) Does there exists a sequence for which $Part.Lim(A) = \mathbb{Q}$?
(c) Can the subset $A \subset \mathbb{R}$ be open? Prove that $A \cup Part.Lim(A) = \overline{A}$.

(5) Given a topological space X and a subset $A \subset X$ with $\mathcal{T}_A = \mathcal{T}_X|_A$, prove: $\underset{\substack{U \subseteq A \\ U \in \mathcal{T}_A}}{\cup} = A$ and $\underset{\substack{A \subseteq V \\ V \text{ is closed in } \mathcal{T}_A}}{\cap} = A$.
Does this imply $Int(A) = A = \overline{A}$? (Check the definition of $Int(A)$, $\overline{A}$.)

(6) Read (and learn): $https://en.wikipedia.org/wiki/Torus$ (well, at least some parts of it :)

(7) Recall that the boundary of a subset $A \subset X$ is defined as $\partial(A) = \overline{A} \setminus Int(A)$.
(a) Find the boundary of (a, b), $[a, b]$, $\mathbb{Z}$, $\mathbb{Q}$ (as subsets of $\mathbb{R}$ with the standard topology)
(b) Prove that $\partial(A) = \overline{A} \cap \overline{X \setminus A}$ and $\partial(A) = \partial(X \setminus A)$.
(c) Prove that $Int(A) \cap \partial(A) = \varnothing$ and $\overline{A} = Int(A) \coprod \partial(A)$.
(d) Prove that $\partial(A) = \varnothing$ iff A is both open and closed.
(e) Prove that A is open iff $\partial(A) = \overline{A} \setminus A$.

(8) Describe the limit points, the boundary, the interior and the closure of the graph of function $f(x) = \frac{sin(\frac{1}{sin(\pi x)})}{sin(\pi x)}$.

(9) Define the Cantor subset of $\mathbb{R}$ by $K := \Big\{ \sum\limits_{n \geq 1} \frac{a_n}{3^n} |\ a_n \in \{0, 2\} \Big\}$.
(a) Prove that $K \subseteq [0, 1]$. Prove that $K \subseteq [0, 1] \setminus (\frac{1}{3}, \frac{2}{3})$. Prove that K does not intersect any segment $(\frac{3s+1}{3^k}, \frac{3s+2}{3^k})$, here $s, k \in \mathbb{N}$.
(b) Present K in the form $[0, 1] \setminus \big(\cup U_i\big)$, where U_i are some open segments.
(c) Prove that K is closed. Describe its boundary, interior, limit points.
(d) Prove that K is uncountable. Can you draw it?

(10) Prove that there is infinity of prime numbers in $\mathbb{N}$, as follows.
(a) Prove that all the possible arithmetic progressions (with infinite number of elements) form a basis for some topology on $\mathbb{N} = \{1, 2 \dots\}$.
(b) For any fixed $d > 0$ consider the progressions $\{i, i + d, i + 2d, \dots\}$, where $i = 1, \dots, d$. Prove that each of them is a closed subset of $\mathbb{N}$. (Because they cover $\mathbb{N}$.)
(c) In particular, for any prime p the progression $\{p, p + 2p, \dots\}$ is a closed subset of $\mathbb{N}$, and these sets cover $\mathbb{N} \setminus \{1\}$. Achieve from here that the amount of prime numbers cannot be finite.

(11) (a) Prove that for the maps $\mathbb{R}^n \stackrel{f}{\to} \mathbb{R}^m$ the topological continuity coincides with the continuity of Calculus.
(b) Fix a continuous function $\mathbb{R} \supseteq \mathcal{D}_f \stackrel{f}{\to} \mathbb{R}$, here $\mathcal{D}_f$ is the domain of f, with $\mathcal{T}_{\mathcal{D}_f} = \mathcal{T}_{\mathbb{R}}|_{\mathcal{D}_f}$. Take its graph, $\Gamma_f = \{(x, f(x))\} \subset \mathcal{D}_f \times \mathbb{R} \subseteq \mathbb{R}^2$, with the induced topology, $\mathcal{T}_{\Gamma_f} = \mathcal{T}_{\mathbb{R}^2}|_{\Gamma_f}$. Prove that the projection to the $\hat{x}$-axis, $\Gamma_f \stackrel{\pi_x}{\to} \mathcal{D}_f$, is a homeomorphims. (Where is the continuity of f used?)
(c) Is the projection onto the $\hat{y}$-axis, $\Gamma_f \stackrel{\pi_y}{\to} Im(f)$, a continuous map? a homeomorphism? (Here $Im(f)$ is the image of f.)

(12) Suppose X has two topologies, $\mathcal{T}_1$, $\mathcal{T}_1$. Prove that the identity map, $(X, \mathcal{T}_1) \stackrel{Id}{\to} (X, \mathcal{T}_2)$, is continuous iff $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Prove that Id is a homeomorphism iff $\mathcal{T}_1 = \mathcal{T}_2$.

(13) Given topological spaces, $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$.
(a) Assume the topology $\mathcal{T}_X$ is trivial. Which functions $X \stackrel{f}{\to} Y$ are continuous? Which functions $Y \stackrel{f}{\to} X$ are continuous?
(b) Assume the topology $\mathcal{T}_X$ is discrete. Which functions $X \stackrel{f}{\to} Y$ are continuous? Which functions $Y \stackrel{f}{\to} X$ are continuous?