Introduction to Topology, 201.1.0091
Homework 3
Spring 2016 (D.Kerner)

(1) Prove
  (a) The composition of continuous maps is continuous.
  (b) If $X \stackrel{f}{\to} Y$ is continuous then for any $A \subset X$ the function $A \stackrel{f|_A}{\to} Y$ is continuous (for $\mathcal{T}_A \stackrel{\varnothing}{=} \mathcal{T}_X|_A$).
  (c) If $X \stackrel{f}{\to} Y$ is continuous and $f(X) \subset B \subset Y$ then the function $X \stackrel{f}{\to} B$ is continuous for $\mathcal{T}_B = \mathcal{T}_Y|_B$.
  (d) If $X \stackrel{f}{\to} Y$ is continuous and $Y \subset Z$, with $\mathcal{T}_Y = \mathcal{T}_Z|_Y$ then the function $X \stackrel{f}{\to} Z$ is continuous.
  (e) $X \stackrel{f}{\to} Y$ is continuous iff there exists an open covering $X = \bigcup\limits_i U_i$ such that $U_i \stackrel{f|_{U_i}}{\to} Y$ is continuous for each i.
  (f) $X \stackrel{f}{\to} Y$ is continuous iff it is continuous at each point of X.

(2) (a) Given a map of topological spaces, $X \stackrel{f}{\to} Y$, prove that the following conditions are equivalent.
  i. f is continuous. ii. For every closed $B \subset Y$ the set $f^{-1}(B) \subset X$ is closed.
  iii. For any $A \subset X$ holds $f(\overline{A}) \subseteq \overline{f(A)}$. iv. For any $B \subset Y$ holds: $f^{-1}(\overline{B}) \supseteq \overline{f^{-1}(B)}$.
  (b) Which of the following properties are implied by continuity? Do they imply continuity?
  i. For any $A \subset X$ holds $f(Int(A)) \supseteq Int(f(A))$. ii. For any $B \subset Y$ holds $Int(f^{-1}(B)) \supseteq f^{-1}(Int(B))$.

(3) A map $X \times Y \stackrel{f}{\to} Z$ is called 'continuous in each variable separately' if for each $y_0 \in Y$ the map $X \stackrel{f(*,y_0)}{\to} Z$ is continuous and for each $x_0 \in X$ the map $Y \stackrel{f(x_0,*)}{\to} Z$ is continuous.
  (a) Show that if f is continuous then it is continuous in each variable separately.
  (b) Give examples (e.g. from Calculus 2,3) of functions continuous in each variable separately but not continuous.

(4) Given a top.space $(Y, \mathcal{T}_Y)$ and a subset $X \subset Y$, prove:
  (a) The emedding map $X \stackrel{i}{\hookrightarrow} Y$ is continuous for the induced topology $\mathcal{T}_X = \mathcal{T}_Y|_X$.
  (b) $\mathcal{T}_Y|_X$ is the coarsest topology for which i is continuous.

(5) (a) Prove that homeomorphism of topological spaces is an equivalence relation.
  (b) Suppose $X \stackrel{f}{\to} Y$ is a homeomorphism of topological spaces. Prove that for any $A \subset X$ holds:
  $f(\overline{A}) = \overline{f(A)}$, $f(Int(A)) = Int(f(A))$, $f(\partial(A)) = \partial f(A)$.

(6) Prove the homeomorphisms in the following cases (with the induced topology):
  i. $\mathbb{R}^2 \approx \{(x,y)|\ x^2+y^2<10\} \approx \{(x,y)|\ |x|<5, |y|<7\} \approx (0,1)\times\mathbb{R} \approx \{(x,y)|\ x>0\} \approx \mathbb{R}^2 \setminus \{(x,0)|\ x\le 0\}$.
  ii. $\{x^2+y^2\le 1\} \approx \{|x|\le 1,\ |y|\le 1\} \approx \{x\ge 0,\ y\ge 0,\ x+y\le 1\}$. iii. $S^2 \setminus \{point\} \approx \mathbb{R}^2$.

(7) Prove the homeomorphisms (here $\mathbb{R}^0$ =a point, S^0 =two points):
  $\mathbb{R}^2 \setminus \mathbb{R}^0 \approx S^1 \times \mathbb{R}$. $\mathbb{R}^3 \setminus \mathbb{R}^0 \approx S^2 \times \mathbb{R}$. $\mathbb{R}^3 \setminus \mathbb{R}^1 \approx S^1 \times \mathbb{R}^2$. $\mathbb{R}^n \setminus \mathbb{R}^k \approx S^{n-k-1} \times \mathbb{R}^{n+1}$.

(8) Identify $Mat_{n\times n}(\mathbb{R}) \approx \mathbb{R}^{n^2}$ (the space of real valued square matrices). This turns $Mat_{n\times n}(\mathbb{R})$ into a topological space with the 'standard topology'.
  (a) Prove that the functions $Mat_{n\times n}(\mathbb{R}) \stackrel{trace}{\to} \mathbb{R}$ and $Mat_{n\times n}(\mathbb{R}) \stackrel{det}{\to} \mathbb{R}$ are continuous.
  (b) Which of the following subsets of $Mat_{n\times n}(\mathbb{R})$ are open/closed? $GL_n(\mathbb{R}) = \{A|\ det(A)\neq 0\}$,
  $SL_n(\mathbb{R}) = \{A|\ det(A)=1\}$, $GL^+_n(\mathbb{R}) = \{A|\ det(A)>0\}$, $O(n) = \{A|\ AA^t = 1\!\!1_{n\times n}\}$
  (c) Prove that the multiplication map, $Mat_{n\times n}(\mathbb{R}) \times Mat_{n\times n}(\mathbb{R}) \stackrel{(A,B)\to AB}{\longrightarrow} Mat_{n\times n}(\mathbb{R})$, is continuous.
  (d) Prove that the inverse map, $GL_n(\mathbb{R}) \stackrel{A\to A^{-1}}{\longrightarrow} GL_n(\mathbb{R})$, is a homeomorphism.

(9) (a) Given a continuous function $\mathbb{R} \supseteq \mathcal{D}_f \stackrel{f}{\to} Im(f) \subseteq \mathbb{R}$ which is a bijection between its domain and its image. Prove that the domain and the graph $\mathcal{D}_f \approx \Gamma_f \subset \mathbb{R}^2$ are homeomorphic (with their embedded topologies).
  (b) Prove: if a continuous function $\mathbb{R} \supset [a,b] \stackrel{f}{\to} Im(f) \subseteq \mathbb{R}$ is bijective onto its image then its inverse is continuous.
  (c) Give an example of a continuous function $\mathbb{R} \supseteq \mathcal{D}_f \stackrel{f}{\to} Im(f) \subseteq \mathbb{R}$ which is bijective onto its image, but whose inverse is not continuous.
  (d) Fix a continuous function $\mathbb{R} \supset [a,b] \stackrel{f}{\to} C \subset \mathbb{R}^n$, bijective onto C (the later is called 'a parameterized continuous curve'). Prove that f is a homeomorphism between $[a,b] \subset \mathbb{R}$ and $C \subset \mathbb{R}^n$.
  (e) Does the last statement hold when the domain is open, $\mathbb{R} \supset (a,b) \stackrel{f}{\to} C \subset \mathbb{R}^n$?

(10) Fix some topological spaces $\{X_j\}_{j\in J}$, here J is an infinite set. Which of the following statements hold for the product topology on $\prod\limits_{j\in J} X_j$? For the box topology on $\prod\limits_{j\in J} X_j$?
  (a) If the subsets $\{A_j \subseteq X_j\}$ are closed/open then the subset $\prod\limits_{j\in J} A_j \subseteq \prod\limits_{j\in J} X_j$ is closed/open.
  (b) $\prod\limits_{j\in J_1 \coprod J_2} X_j = \Big(\prod\limits_{j\in J_1} X_j\Big) \times \Big(\prod\limits_{j\in J_2} X_j\Big)$. $\prod\limits_{j\in J} \overline{A_j} = \overline{\prod\limits_{j\in J} A_j}$. $\prod\limits_{j\in J} Int(A_j) = Int(\prod\limits_{j\in J} A_j)$.