Introduction to Topology, 201.1.0091

Homework 4

Spring 2016 (D.Kerner)

(1) (a) Prove that a subspace of a Hausdorff space is Hausdorff.
(b) If $(X, \mathcal{T}_X)$ is Hausdorff then any refinement of $\mathcal{T}_X$ is Hausdorff.
(c) Prove that the property of being Hausdorff is preserved under homeromorphisms. $\varnothing$
(d) Prove that the diagonal map $X \xrightarrow{\Delta} X \times X$, $\Delta(t) = (t,t)$ is continuous and X is Hausdorff iff $\Delta(X)$ is closed in $X \times X$.
(e) Prove that any product of Hausdorff spaces is Hausdorff (both in the box and the product topologies).

(2) Given an infinite collection of topological spaces, $\{X_\alpha\}$, prove that $\mathcal{T}^{box}_{\prod\limits_\alpha X_\alpha} = \mathcal{T}^{product}_{\prod\limits_\alpha X_\alpha}$ iff the topologies on X_α are trivial except for a finite number of values of α.

(3) Prove that the projections $\prod\limits_\alpha X_\alpha \xrightarrow{\pi_\beta} X_\beta$ are open maps, i.e. the image of an open set is open. (Both in $\mathcal{T}^{box}_{\prod\limits_\alpha X_\alpha}$ and in $\mathcal{T}^{product}_{\prod\limits_\alpha X_\alpha}$.) Are the projections always closed maps (i.e. the image of a closed set is closed)?

(4) (a) Prove that the following functions $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{d_i} \mathbb{R}_{\geq 0}$ define metrics on $\mathbb{R}^n$:
i. $d_p(x,y) = \sqrt[p]{\sum\limits_{i=1}^{n} |x_i - y_i|^p}$, for $p \in \mathbb{N}$. (Use the inequality of Minkowski.) ii. $d_\infty(x,y) = \max\limits_{i=1,\dots,n} \Big(|x_i - y_i|\Big)$.
(b) For any fixed $x, y \in \mathbb{R}^n$ prove: $\lim\limits_{p\to\infty} d_p(x,y) = d_\infty(x,y)$.
(c) What are the unit balls centered at the origin for these metrics, for $p = 1, 2, \infty$?
(d) For any $\infty \geq p > q \geq 1$ and any $r > 0$ prove: $Ball^{(d_p)}_{\frac{r}{\sqrt[p]{n}}}(0) \subsetneq Ball^{(d_q)}_r(0) \subsetneq Ball^{(d_p)}_r(0)$.
(e) Prove that all the metrics d_p (for all the $1 \leq p \leq \infty$) define the same topology on $\mathbb{R}^n$.

(5) (a) Suppose X has two metrics, d_1, d_2. Which of the following functions are metrics?
i. $d_1 + d_2$, ii. $max(d_1, d_2)$, iii. $min(d_1, d_2)$, iv. $d_1 d_2$, v. $\frac{d_1}{1+d_1}$, vi. $min(d_1, 1)$
(b) Given a metric space (X,d), and a function $\mathbb{R} \xrightarrow{f} \mathbb{R}$, prove that $X \times X \xrightarrow{f(d(x,y))} \mathbb{R}$ is also a metric if f satisfies the following: i. $f(0) = 0$, ii. $f(x)$ is increasing, iii. $f(a+b) \leq f(a) + f(b)$ for any $a, b \in \mathbb{R}_+$.
(c) Given a metric space (X,d) prove that the following functions are metrics and they all define the same topology on X: i. $\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$, ii. $\bar{d}(x,y) = max(d(x,y), 1)$.

(6) (a) Prove that the metric topology is always Hausdorff.
(b) Let $(X, \mathcal{T}_X)$ be a finite set with some topology. Prove that $\mathcal{T}_X$ is metrizable iff it is discrete.
(c) Given a metric space (X, d_X) and a subset $A \subset X$ define the induced metric on A by restriction, $d_A(x,y) := d_X(x,y)$. (Prove that this is a metric.) Prove that the metric topology on (A, d_A) coincides with the induced topology $\mathcal{T}_X|_A$.

(7) Fix a metric space (X,d). The distance between two subsets $A, B \subset X$ is defined as $d(A,B) := \inf\limits_{\substack{x\in A\\ y\in B}} d(x,y)$.
(a) Prove that $x \in \overline{A}$ iff $d(x,A) = 0$.
(b) Find two disjoint closed subsets of $\mathbb{R}^1$ which are zero distance apart. Find two disjoint curves in $\mathbb{R}^2$ which are zero distance apart. (Curves=images of either of $[0,1]$, $(0,1)$, $(0,1]$, $[0,1)$ under a continuous map.)

(8) (a) Given a metric space (X,d), prove that every metrically closed ball is topologically closed.
(b) Give examples of metric spaces in which there exists a metrically closed ball (of positive radius) that is topologically open. Give an example of space in which every metrically closed ball is topologically open.
(c) Give an example of a metric space, (X,d), in which there exist two balls, say B_1, B_2, such that $B_1 \subsetneq B_2$, but the radii 'suggest' the converse: $r(B_1) > r(B_2)$. (Give example(s) with $|B_1|$ as small as possible, $|B_2|$ as large as possible and $\frac{r(B_1)}{r(B_2)}$ close to 2.)
(d) Prove that for any metric space, (X,d), any points $x_1, x_2 \in X$ and any number $r > d(x_1,x_2)$ the inclusion holds: $B_{r-d(x_1,x_2)}(x_1) \subseteq B_r(x_2)$.

(9) (a) Fix $p \in \mathbb{N}$ and let $l_p = \Big\{\{a_n\}_{n=1}^\infty \,|\, \sum\limits_{n=1}^{\infty} |a_n|^p < \infty\Big\} \subset \mathbb{R}^\mathbb{N}$. Prove that l_p is a vector subspace of $\mathbb{R}^\mathbb{N}$.
(b) Prove that $d_p(x,y) := \sqrt[p]{\sum\limits_{n=1}^{\infty} |x_n - y_n|^p}$ is a metric on l_p.
(c) Compare the topologies on $l_p \subset \mathbb{R}^\mathbb{N}$ induced from the box/product/uniform topologies on $\mathbb{R}^\mathbb{N}$ and the topology induced by $d_p(x,y)$. (Which are finer/coarser?)
(d) Are the subsets $X_n = \{(x_1, x_2, \dots) |\ x_i = 0\ for\ i > n\}$ open/closed? (In which topology?)
(e) Is the subset $\bigcup\limits_{n\geq 1} X_n$ open/closed?
(f) What about the Hilbert cube $H = \prod\limits_{n\in\mathbb{N}} [0, \frac{1}{n}]$?

(10) Denote by l_0 the subset of all the sequences (of real numbers) which are eventually zero. Prove that $l_0 \subset \mathbb{R}^\mathbb{N}$ is a vector subspace. What are the closure, $\overline{l_0}$, and the interior, $Int(l_0)$, in the box/product/uniform topologies?