Introduction to Topology, 201.1.0091

Homework 5

Spring 2016 (D.Kerner)

(1) (a) For which of the product/box/uniform topologies on $\prod\limits_{i\in\mathbb{N}}\mathbb{R}$ the following functions are continuous? i. $f(t)=(t,t,t,\dots)$, ii. $f(t)=(t,2t,3t,\dots)$, iii. $f(t)=(t,\frac{t}{2},\frac{t}{3},\dots)$.

(b) Prove that $\mathcal{T}^{product}_{\prod\limits_\alpha X_\alpha}$ is the coarsest topology for which all the projections $\prod\limits_\alpha X_\alpha\stackrel{\pi_\alpha}{\to}X_\alpha$ are continuous. Prove that in $\mathcal{T}^{product}_{\prod\limits_\alpha X_\alpha}$ the continuity of $X\stackrel{\prod\limits_\alpha f_\alpha}{\to}\prod\limits_\alpha X_\alpha$ implies the continuity of each f_α.

(c) Suppose $X_\alpha=X$, for all α. Suppose some topology $\mathcal{T}_{\prod\limits_\alpha X_\alpha}$ is prescribed and some open set $\prod\limits_\alpha\mathcal{U}_\alpha\in\mathcal{T}_{\prod\limits_\alpha X_\alpha}$ satisfies: $\{\mathcal{U}_\alpha\subseteq X_\alpha\}_\alpha$ are open but $\bigcap\limits_\alpha\mathcal{U}_\alpha\subset X$ is not open. Prove that for this topology the 'multi-identity' map, $X\stackrel{\prod\limits_\alpha f_\alpha}{\to}\prod\limits_\alpha X_\alpha$, $f_\alpha(t)=t$, is not continuous.

(d) Suppose a topology $\mathcal{T}_{\prod\limits_{\mathbb{N}}\mathbb{R}}$ contains the following open set: $\prod\limits_i(a_i,b_i)$, here all $\{a_i\},\{b_i\}$ are finite. Give an example of continuous functions $\{\mathbb{R}\stackrel{f_i}{\to}\mathbb{R}\}_{i\in\mathbb{N}}$ such that the function $\mathbb{R}\stackrel{\prod\limits_i f_i}{\to}\prod\limits_i\mathbb{R}$ is not contininuous.

(2) (a) For which of the topologies $\mathcal{T}^{product}_{\prod\limits_{\alpha\in\mathbb{N}}\mathbb{R}}$, $\mathcal{T}^{box}_{\prod\limits_{\alpha\in\mathbb{N}}\mathbb{R}}$, $\mathcal{T}^{uniform}_{\prod\limits_{\alpha\in\mathbb{N}}\mathbb{R}}$ the following sequences converge? i. $x_i=(\underbrace{0,\dots,0}_{i},i,i,\dots)$, ii. $y_i=(\underbrace{0,\dots,0}_{i},\frac{1}{i},\frac{1}{i},\dots)$, iii. $z_i=(\underbrace{\frac{1}{i},\dots,\frac{1}{i}}_{i},0,\dots,0,\dots)$, iv. $w_i=(\underbrace{\frac{1}{i},\frac{1}{i}}_{2},0,\dots,0,\dots)$.

(b) Prove that in $\mathcal{T}^{product}_{\prod\limits_{\alpha\in I}X_\alpha}$ holds: the product converges, $\{x^{(\alpha)}_n\}_{\substack{\alpha\in I\\ n=1,2,\dots}}\to\{x^{(\alpha)}\}_{\alpha\in I}$ iff each coordinate sequence converges, $\{x^{(\alpha)}_n\}_{n=1,2,\dots}\to x^{(\alpha)}$, for each α.

(c) Suppose a topology $\mathcal{T}_{\prod\limits_{\alpha\in I}X_\alpha}$ contains at least one set $\prod\limits_\lambda\mathcal{U}_\alpha$, with $\varnothing\neq\mathcal{U}_\alpha\neq X_\alpha$ for infinite number of values of α. Construct a sequence $\{x^{(\alpha)}_n\}_{n\in\mathbb{N}}$ which converges in each coordinate separately, but not as a sequence in $\mathcal{T}_{\prod\limits_{\alpha\in I}X_\alpha}$.

(3) (a) Fix a collection of metric spaces, $\{(X_\alpha,d_\alpha)\}$, prove the comparison of the product, uniform and box topologies: $\mathcal{T}^{product}_{\prod\limits_\alpha X_\alpha}\subseteq\mathcal{T}^{uniform}_{\prod\limits_\alpha X_\alpha}\subseteq\mathcal{T}^{box}_{\prod\limits_\alpha X_\alpha}$. Give examples with the proper inclusions.

(b) Given a countable collection of metric spaces, $\{(X_i,d_i)\}_{i\in\mathbb{N}}$, prove that the following functions define metrics on $\prod\limits_{i\in\mathbb{N}}X_i$, and the corresponding topology is precisely the product topology:

i. $d(x,y)=\sum\limits_{i=1}^{\infty}\frac{1}{2^i}\frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}$, ii. $d(x,y)=sup\left\{\frac{\overline{d_i}(x_i,y_i)}{i}\right\}$, here $\overline{d_i}(x_i,y_i)=min(d_i(x_i,y_i),1)$.

(4) (a) Given a metric space (X,d_X) and a surjective map $X\stackrel{f}{\to}Y$, which of the following always defines a metric on Y? $f_*(d_X)(y_1,y_2)=\sup\limits_{\substack{f(x_1)=y_1\\ f(x_2)=y_2}}d_X(x_1,x_2)$, $f_*(d_X)(y_1,y_2)=\inf\limits_{\substack{f(x_1)=y_1\\ f(x_2)=y_2}}d_X(x_1,x_2)$.

(b) Given a metric space (Y,d_Y) and an injective map $X\stackrel{f}{\to}Y$, does the function $f^*(d_Y)(x_1,x_2)=d_Y(f(x_1),f(x_2))$ always define a metric on X?

(c) Given two homeomorphic spaces, $X\approx Y$, with $\mathcal{T}_X$ metrizable, prove that $\mathcal{T}_Y$ is metrizable as well. (In particular one gets: the spaces $(\mathbb{R}^{\mathbb{N}},\mathcal{T}^{product}_{\mathbb{R}^{\mathbb{N}}})$, $(\mathbb{R}^{\mathbb{N}},\mathcal{T}^{box}_{\mathbb{R}^{\mathbb{N}}})$, are not homeomorphic.)

(5) (a) Fix an infinite set X, with the topology of finite complements. Fix a sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ with pairwise distinct elements, i.e.: $x_i\neq x_j$ for $i\neq j$. Prove that $\{x_n\}$ converges to <u>any</u> point of X in this topology.

(b) Take $\mathbb{R}$ with the topology of finite or countable complements. (This is a refinement of the topology of finite complements on $\mathbb{R}$.) What are the convergent sequences in this topology?

(6) (a) Given a metric space (X,d) and a sequence $\{x_n\}$. Prove: $x_n\to x$ iff $d(x_n,x)\to 0$.

(b) Suppose $x_n\to x$ and $y_n\to y$. Prove that $d(x_n,y_n)\to d(x,y)$.