Introduction to Topology, 201.1.0091
Homework 6
Spring 2016 (D.Kerner)

(1) (a) Prove: if X is connected and the map $X \stackrel{f}{\to} Y$ is continuous then $f(X) \subseteq Y$ is connected. In particular, if $X \stackrel{homeo}{\approx} Y$ then X is connected iff Y is connected.
(b) Prove that the following spaces are pairwise non-homeomorphic (all the topologies are the usual ones): i. $(0,1)$, ii. $[0,1]$, iii. $[0,1] \cup [3,4]$, iv. $[-1,0) \cup (0,1]$, v. S^1, vi. $\mathbb{R}^n$, $n > 1$.
(c) Prove that S^1 is not homeomorphic to any subset of $\mathbb{R}$.
(d) Prove that for any continuous function $S^1 \stackrel{f}{\to} \mathbb{R}$ any point of the set $f(S^1) \setminus \{inf(f(S^1)), sup(f(S^1))\}$ has at least two preimages in S^1.
(e) Classify the alphabet letters up to homeomorhism: A,B,C,D,E,F,G,. . .

(2) (a) Prove that every infinite set is connected in the topology of finite complements.
(b) Show that a finite Hausdorff space is totally disconnected, i.e. its only connected subsets are the one-point sets.
(c) Show that X is connected iff for any $\varnothing \neq A \subsetneq X$ holds: $\partial(A) \neq \varnothing$.
(d) Show that $A \subset X$ is non-connected iff exists $X_1, X_2 \subset X$ satisfying: $A = X_1 \coprod X_2$, $\overline{X_1}^{(X)} \cap X_2 = \varnothing$, $X_1 \cap \overline{X_2}^{(X)} = \varnothing$. (Note that here X_1, X_2 are not necessarily open. Does the statement hold with the condition $\overline{X_1}^{(X)} \cap \overline{X_2}^{(X)} = \varnothing$?)
(e) Prove that if $A \subset X$ is connected and $A \subseteq B \subseteq \overline{A}$ then B is connected. Does the converse hold?
(f) Suppose both $A \cup B$ and $A \cap B$ are connected. Does this imply the connectedness of A, B?

(3) (a) Given a continuous map $X \stackrel{f}{\to} Y$, with X connected, prove that the graph $\Gamma_f \subseteq X \times Y$ is a connected subset.
(b) Prove that if a connected subset $C \subsetneq X$ intersects both $A \subsetneq X$ and $X \setminus A$, then C intersects ∂A. (Compare this to the intermediate value theorem of calculus.)

(4) (a) Given X with two topologies, $\mathcal{T}_1 \subsetneq \mathcal{T}_2$. Show that if $(X, \mathcal{T}_2)$ is connected then $(X, \mathcal{T}_1)$ is connected as well. Give examples where the converse does not hold.
(b) Prove that $(\prod_\alpha X_\alpha, \mathcal{T}^{product}_{\prod_\alpha X_\alpha})$ is connected iff all X_α are connected.
(c) Suppose X is connected, while Y has two connected components, $Y = \mathcal{U}_1 \coprod \mathcal{U}_2$. Suppose in the topology $\mathcal{T}_{X\times Y} \subsetneq \mathcal{T}^{product}_{X\times Y}$ at least one of $X \times \mathcal{U}_1$, $X \times \mathcal{U}_2$ is not open. Prove that $(X \times Y, \mathcal{T}_{X\times Y})$ is connected.
(d) Prove that $(\prod_{i\in\mathbb{N}} \mathbb{R}, \mathcal{T}^{box}_{\prod_{i\in\mathbb{N}} \mathbb{R}})$ is not connected. (Check the subset {bounded sequences}$\subset \prod_{i\in\mathbb{N}} \mathbb{R}$ and its complement.)
(e) What about $\prod_{i\in\mathbb{N}} \mathbb{R}$ in the uniform topology?

(5) (a) Split $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ into the union of connected components.
(b) Prove that an open subset of $\mathbb{R}^n$ can have at most countably many connected components.
(c) X is called totally disconnected if its only connected components are points. Suppose $X \subset \mathbb{R}^n$ is totally disconnected, does this mean that $\mathbb{R}^n \setminus X$ contains an open subset of $\mathbb{R}^n$?

(6) Check the (path-)connectedness in the following cases.
(a) i. S^n with a finite number of punctures (what about $n = 0$?). ii. $\mathbb{R}^n$ with a countable number of punctures.
iii. $\mathbb{R}^{n+1} \supset S^n \setminus S^k$, here the embedding $S^k \subset S^n$ is equatorial, i.e. $S^k = \{\sum_{i=0}^{k} x_i^2 = 1,\ x_{i>k} = 0\} \subset \mathbb{R}^{n+1}$.
(b) i. $\mathbb{Q}^2 \subset \mathbb{R}^2$, ii. $(\mathbb{Q} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Q}) \subset \mathbb{R}^2$,
(c) The "squeezed comb" space: $X = ([0,1] \times 0) \cup (K \times [0,1]) \cup (0 \times [0,1])$, where $K = \{\frac{1}{n} |\ n \in \mathbb{N}\}$.
(d) The deleted comb space: $X = ([0,1] \times 0) \cup (K \times [0,1]) \cup (0 \times [0,1]) \setminus (0 \times (0,1))$, where $K = \{\frac{1}{n} |\ n \in \mathbb{N}\}$.

(7) (a) Suppose all X_α are path-connected, is $(\prod_\alpha X_\alpha, \mathcal{T}^{product}_{\prod_\alpha X_\alpha})$ necessarily path connected?
(b) Suppose $A \subset X$ is (path-)connected. Are the sets $Int(A)$, $\overline{A}$, $\partial(A)$ necessarily (path-)connected?
(c) Suppose $X \stackrel{f}{\to} Y$ is continuous and X is path-connected. Is $f(X) \subseteq Y$ path connected?

(8) (a) Prove that a pancake (i.e. an open bounded subset of $\mathbb{R}^2$, not necessarily connected) can be cut into two pieces of precisely the same area by one (straight) cut.
(b) Prove that two pancakes (i.e. two open bounded subsets of $\mathbb{R}^2$) can be simultaneously cut into halves of (accordingly) equal area by one (straight) cut. (Hint: you need to use that some relevant subset is conected. You should represent this subset as $f^{-1}(..)$ for some continuous function.)

(9) Let $p_d(x,y)$ be a polynomial in x, y, of total degree d. How many (path-)connected components can have the space $\mathbb{R}^2 \setminus \{p_2(x,y) = 0\}$? Give example of $p_d(x,y)$ for which $\mathbb{R}^2 \setminus \{p_d(x,y) = 0\}$ has $\binom{d+1}{2} + 1$ connected components. (Hint: take d lines.) It is also worth to read the wiki-page on *Arrangements of hyperplanes*.

(10) Classify the (path-)connected components of the following matrix subspaces of $Mat_{n\times n}(\mathbb{R})$ and $Mat_{n\times n}(\mathbb{C})$:
i. $Mat^{sym}_{n\times n}(\mathbb{R}) = \{A |\ A = A^t\}$, ii. $Mat^{herm}_{n\times n}(\mathbb{C}) = \{A |\ A^t = \overline{A}\}$, iii. $GL(n, \mathbb{R})$, iv. $O(n) = \{A |\ A \cdot A^t = \mathbb{1}_{n\times n}\}$, v. $SO(n) = \{A \in O(n) |\ det(A) = 1\}$, vi. $GL(n, \mathbb{C})$, vii. $Mat^{sym}_{n\times n}(\mathbb{R}) \cap GL(n, \mathbb{R})$, viii. $U(n) = \{A |\ A \cdot \overline{A^t} = \mathbb{1}_{n\times n}\}$.
(One way to do this is to turn the discrete steps of the Gauss algorithm into continuous paths.)