Introduction to Topology, 201.1.0091

Homework 7

Spring 2016 (D.Kerner)

(1) Prove that a connected metric space having more than one point is uncountable.

(2) The spiral $C \subset \mathbb{R}^2$ is defined in the polar coordinates by $\{r = \frac{1}{(1+\phi)^\alpha},\ \phi \in [0,\infty)\}$, here $\alpha > 0$ is a constant. Is $\overline{C} \subset \mathbb{R}^2$ path connected? Does the answer depend on α, i.e. on the total length of C?

(3) (a) Which spaces with discrete topology are compact?
   (b) Prove that any space with the topology of finite complements is compact.

(4) (a) Given $(X, \mathcal{T}_X)$ and a subset $A \subset X$, prove that A is a compact subset iff $(A, \mathcal{T}_X|_A)$ is a compact space.
   (b) Given X with topologies $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Does the compactness of $(X, \mathcal{T}_1)$ imply that of $(X, \mathcal{T}_2)$? Or the converse?
   (c) Prove that finite unions of compact subsets are compact.

(5) (a) Prove that a closed subset in a compact space is compact. Prove that every compact subspace of a Hausdorff space is closed.
   (b) Prove that a compact subset of a metric space is bounded and closed.
   (c) Give an example of a closed, bounded but non-compact subset of a metric space.
   (d) Prove that arbitrary intersections of compact subsets of a Hausdorff space are compact.
   (e) Give an example of a (non-Hausdorff) space with compact subsets A, B, such that $A \cap B$ is non-compact.

(6) (a) Suppose X is compact, while Y is Hausdorff. Prove that any continuous map $X \stackrel{f}{\to} Y$ is closed, i.e. f sends closed subsets of X to closed subsets of Y.
   (b) In particular, prove that if $X \stackrel{f}{\to} Y$ is a continuous bijection, X is compact, Y is Hausdorff, then f is a homeomorphism. (i.e. in this case there is no need to check the continuity of f^{-1})
   (c) Prove that the projection $X \times Y \stackrel{\pi_X}{\to} X$ is a closed map if Y is compact. (What happens if Y is non-compact?)
   (d) Suppose Y is compact and Hausdorff. Prove that $X \stackrel{f}{\to} Y$ is continuous iff the graph $\Gamma_f \subset X \times Y$ is a closed subset.

(7) Let $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. Which of the following subspaces of $Mat_{n\times n}(\Bbbk)$ are compact? (for the standard topology on $Mat_{n\times n}(\Bbbk)$.) $GL(n,\Bbbk)$, $SL(n,\Bbbk) = \{A \in Mat_{n\times n}(\Bbbk) |\ det(A) = 1\}$, $O(n)$, $U(n)$.

(8) (a) Given a continuous map $(X, \mathcal{T}_X) \stackrel{f}{\to} (Y, \mathcal{T}_Y)$, prove that if X is compact then $f(X)$ is compact. Does the converse hold?
   (b) Prove that if $(X, \mathcal{T}_X)$ is compact and $(X, \mathcal{T}_X) \stackrel{f}{\to} \mathbb{R}$ is continuous then f achieves its minimum and maximum on X.

(9) (a) Let $\varnothing \neq X_1, X_2 \subset X$ be two disjoint compact subsets of a Hausdorff space. Prove that they can be separated by open subsets, i.e. $X_i \subset \mathcal{U}_i$, $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$.
   (b) (A generalization of the tube lemma) Given some subsets $A \subset X$, $B \subset Y$ and an open set neighborhood $\mathcal{U}$ of $A \times B$, i.e. $A \times B \subseteq \mathcal{U} \subseteq X \times Y$. Prove: if A, B are compact then exist some open sets $A \subseteq \mathcal{U}_A \subseteq X$, $B \subseteq \mathcal{U}_B \subseteq Y$, satisfying: $A \times B \subseteq \mathcal{U}_A \times \mathcal{U}_B \subseteq \mathcal{U}$.

(10) A closed continuous surjective map $X \stackrel{f}{\to} Y$ is called *perfect* if $f^{-1}(y)$ is compact for any $y \in Y$. Prove that if f is perfect and Y is compact then X is compact as well.

(11) (a) Given a metric space (X, d) and a subset $\varnothing \neq A \subset X$, prove that $d(x, A) = 0$ iff $x \in \overline{A}$.
   (b) Prove that $X \times X \stackrel{d}{\to} \mathbb{R}_{\geq 0}$ is a continuous function.
   (c) Prove that if A is compact then $d(x, A) = d(x, a)$ for some $a \in A$.
   (d) An ϵ-neighborhood of $A \subset X$ is $U(A, \epsilon) = \{x \in X |\ d(x, A) < \epsilon\}$. (It can be covered by open balls $\{Ball_\epsilon(a)\}_{a\in A}$.) Suppose A is compact and $\mathcal{U}$ is an open neighborhood of A. Prove that $\mathcal{U}$ contains some ϵ-neighborhood of A.
   (e) Give examples of closed-but-unbounded or bounded-but-non-closed subsets of $\mathbb{R}^n$ for which (c) does not hold.
   (f) Suppose $A, B \subset X$ are compact and disjoint. Prove that $d(A, B) > 0$.

(12) Let X be compact, Hausdorff. Given a countable family of sets, $\{A_n\}_{n\in\mathbb{N}}$, with empty interiors, $Int(A_n) = \varnothing$, prove that $Int(\cup A_n) = \varnothing$.

(13) Prove that if $U \subset \mathbb{R}^n$ is connected and open then U is path-connected.

(14) A map $X \stackrel{f}{\to} X$ satisfying $d(f(x), f(y)) < d(x, y)$, for any $x, y \in X$, with $x \neq y$, is called a shrinking map. Suppose (X, d) is compact, prove that the equation $f(x) = x$ has unique solution for any shrinking map.

(15) A collection of subsets $\{A_\alpha\}$ of X is called centered (or "satisfies the finite intersection property") if any finite intersection of them is non-empty. Prove that $(X, \mathcal{T}_X)$ is compact iff any centered collection of closed subsets has a non-empty intersection.

(16) Suppose X is limit point compact.
   (a) If $X \stackrel{f}{\to} Y$ is continuous, is $f(X)$ necessarily limit point compact?
   (b) If $A \subset X$ is closed, is $f(A)$ necessarily closed?

(17) Let (X, d) be compact, with an open cover $X = \underset{\alpha}{\cup}\mathcal{U}_\alpha$. Prove that there exists $\epsilon > 0$ such that for any $x \in X$ the open $Ball_\epsilon(x)$ is contained in at least one $\mathcal{U}_\alpha$.