Introduction to Topology, 201.1.0091
Homework 8
Spring 2016 (D.Kerner)

(1) Given a metric space (X, d), define the distance between a point x and a subset $A \subset X$ by $d(x, A) = \inf\limits_{a \in A} d(x, a)$.
   (a) Prove that the function $f(x) = d(x, A)$ is continuous.
   (b) Prove: if A is compact then $d(x, A) = d(x, a)$ for some $a \in A$. Is the condition "A is closed" sufficient for this statement?

(2) Define the ϵ-neighborhood of $A \subset X$ to be $\mathcal{U}(A, \epsilon) = \{x | \ d(x, A) < \epsilon\}$. Suppose A is compact and $A \subset \mathcal{U}_{open}$. Prove that $\mathcal{U}$ contains some ϵ-neighborhood of A. Does this hold when A is just closed, but non-compact?

(3) Let $\{A_n\}$ be a countable collection of closed subsets in a compact Hausdorff space X. Prove: if $Int(A_n) = \varnothing$ for any n, then $Int(\bigcup\limits_n A_n) = \varnothing$. (Hint: imitate the proof of "a compact Hausdorff space must be uncountable".)

(4) (a) Let $l_p = \{x | \ d_p(x) < \infty\} \subset \prod\limits_{n=1}^{\infty} \mathbb{R}$, where $d_p(x, y) = \sqrt[p]{\sum\limits_i |x_i - y_i|^p}$ (for $p = \infty$ we define $d_\infty(x, y) = sup|x_i - y_i|$). Which of l_p are compact? In which of l_p the closed ball is compact? What about the spheres?
   (b) Fix some sequence of real numbers, $a_n \to 0$, and consider $V^{\{a_n\}} = \{x_n | \ |x_n| \le |a_n|\} \subseteq l_\infty$. Is $V^{\{a_n\}}$ compact? Is it connected?

(5) Fix a self-map of a metric space $(X, d) \xrightarrow{f} (X, d)$. The map is called *a shrinking map* if $d(f(x), f(y)) < d(x, y)$ for any $x, y \in X$. The map is called *a contracting map* if $\frac{d(f(x), f(y))}{d(x,y)} < \lambda$ for any $x \neq y \in X$ and some fixed $\lambda < 1$.
   (a) Prove that a shrinking map is contiuous. Give an example of a shrinking map which is not contracting.
   (b) Suppose f is a shrinking map and X is compact. Prove that f has the unique fixed point, i.e. the equation $f(x) = x$ has the unique solution.

(6) (a) Prove that a compact space is limit-point-compact.
   (b) Prove that sequential compactness implies limit point compactness.
   (c) Let $X = [0, 1]^{\mathbb{N}}$ with the uniform topology. Find an infinite subset of X with no limit point.

(7) (a) Describe (up to homemorphism) the one-point compactifications of the following spaces:
   i. X is the disjoint union of n open intervals. ii. $\mathbb{R}^n$. iii. S^2 with two punctures; with n punctures.
   iv. $\mathbb{N}$. v. $\{0 < x^2 + y^2 < 1\} \subset \mathbb{R}^2$. vi. $\{x, y \in [-1, 1], \ |xy| < 1\} \subset \mathbb{R}^2$. vii. $[0, 1] \times \mathbb{R} \subset \mathbb{R}^2$
   (b) Does there exist a (Hausdorff) compactification of $(0, 1]$ for which all the sequences $\{\frac{1}{2\pi n + a}\}$ converge (here $a \in [0, \pi)$ is fixed) and have pairwise distinct limits? (Hint: take the closure of the graph of $(0, 1] \xrightarrow{sin(\frac{1}{x})} \mathbb{R}$.)

(8) Fix a field $\Bbbk$ and consider the ring of (formal) power series in one variable, $\Bbbk[[x]] = \{\sum\limits_{n=0}^{\infty} a_n x^n, \ a_n \in \Bbbk\}$. Fix the maximal ideal, $\mathfrak{m} = (x) \subset \Bbbk[[x]]$, and define the topology $\mathcal{T}_{\Bbbk[[x]]}$: for any $f \in \Bbbk[[x]]$ the basic open neighborhoods are $\Big\{\mathcal{U}_{f,N} := \{f\} + \mathfrak{m}^N\Big\}_N$. (This space shows up constantly in Commutative Algebra, Algebraic Geometry, Singularity Theory etc. The idea behind this topology is: "near" the origin the higher order terms in Taylor's expansion are "small".)
   (a) Prove that $\mathcal{T}_{\Bbbk[[x]]}$ is a topology. Prove that the sets $\mathcal{U}_{f,N}$ are closed. Is $\Bbbk[[x]]$ Hausdorff?
   (b) What are the convergent sequences in this space? Is $\Bbbk[[x]]$ (locally) connected? (locally) compact?
   (c) Prove that $d(f, g) = e^{-ord(f-g)}$ is a metric. Here the Taylor order is defined by $ord(f) = max\{N | \ f \in \mathfrak{m}^N\}$. Prove that $d(f, g)$ is an "'ultrametric"', i.e. for any f, g, h holds: $d(f, g) \le max\Big(d(f, h), d(g, h)\Big)$.
   (d) Prove that $d(*, *)$ induces the topology $\mathcal{T}_{\Bbbk[[x]]}$. What are the open/closed balls?
   (e) Consider the subspace of polynomials, $\Bbbk[x] \subset \Bbbk[[x]]$. Is $\Bbbk[x]$ open/closed inside $\Bbbk[[x]]$? What is $\overline{\Bbbk[[x]]}$?

(9) (a) Prove that the subspace $\mathbb{Q} \subset \mathbb{R}$, with the induced topology, is not locally compact at any point.
   (b) Give an example of two locally compact subspaces of $\mathbb{R}$ whose union is not locally compact.

(10) (a) Let X be Hausdorff. Prove that X is locally compact iff for any $x \in X$ and any (open) neighborhood $x \in \mathcal{U}$ exists a neighborhood $x \in V \subseteq \mathcal{U}$ satisfying: $\overline{V} \subseteq \mathcal{U}$ and $\overline{V}$ is compact.
   (b) Let X locally compact, Hausdorff and let $A \subseteq X$. Prove: if A is locally closed or open then it is locally compact.
   (c) Prove that X is homeomorphic to an open subset of a compact Hausdorff space iff X is locally compact and Hausdorff.

(11) A function $X \xrightarrow{f} \mathbb{R}$ is called locally bounded if for any $x \in X$ exists an open $x \in \mathcal{U}$ such that $f(\mathcal{U})$ is a bounded subset of $\mathbb{R}$. Prove that if X is compact then any locally bounded function is bounded.

(12) Let $\Big\{p_\alpha(\{x_\beta\}) = 0\Big\}_{\substack{\alpha \in A \\ \beta \in B}}$ be an infinite system of polynomial equations. Suppose the field $\Bbbk$ is finite. Use Tychonoff's theorem to prove: if every finite subsystem is solvable then the whole system is solvable.