Introduction to Topology, 201.1.0091

Homework 9

Spring 2016 (D.Kerner)

(1) Linear topology on vector spaces.

Fix a vector space $V_{\Bbbk}$ (of any dimension). An affine subspace of V is the set of the form $\{v\}+W$,

- (a) for some vector $v \in V$ and some subspace $W \subseteq V$. Prove that any (non-empty) intersection of affine subspaces is an affine subspace, and similarly for any sum of affine subspaces.
- (b) A Hausdorff topology on V is called linear if any translations of open sets are open and the local basis (of open neighborhoods) at the origin consists of some linear subspaces. Prove:
  - (i) Any isomorphism of vector spaces preserves the linearity of topology.
  - (ii) The induced topology on any vector subspace is linear.
  - (iii) For any product of vector spaces the product/box topologies are linear.
  - (iv) The closure of any vector/affine subspace of V is a vector/affine subspace. In particular, if an affine subspace is open then it is also closed.
  - (v) A finite dimensional vector space with linear topology is discrete (i.e. the topology is discrete).
  - (vi) Any discrete vector space has a linear topology.
- (c) A vector space with linear topology is called *linearly compact* if for any family of closed affine subspaces, with the property of non-empty finite intersections, the total intersection is non-empty.
  - (i) Prove that the linear compactness is preserved under homemorphisms which are isomorphisms of vector spaces.
  - (ii) V with the discrete (and hence linear) topology is linearly compact iff $dim(V) < \infty$.
  - (iii) If V is linearly compact then any closed subspace of V is linearly compact.
  - (iv) Any product of linearly compact vector spaces is linearly compact. (The proof goes along the same lines as the proof of Tychonoff's theorem.)
- (d) Given a system of linear equations, $\{l_\alpha(\{x_\beta\}) = 0\}_{\substack{\alpha\in A\\ \beta\in B}}$, (any number of equations, in any number of variables, over any field $\Bbbk$). Prove that the system is solvable iff every finite subsystem of it is solvable.

(2) Let $G = (V, E)$ be an undirected graph (V-the set of vertices, E-the set of edges). A k-colouring of G is a function $V \overset{col}{\to} \{1, 2, \dots, k\}$ such that if $v, v' \in E$ then $col(v) \neq col(v')$.. A graph is called k-colourable if it admits a k-colouring. Prove that if every finite subgraph of G is k-colourable then G is k-colourable. (Use Tychonoff's theorem.)

(3)
- (a) Find the "simplest" metrizable compactifications of $[-1, 0) \cup (0, 1]$ for which the folowing functions extend in a continuous manner: i. $f(x) = \frac{x}{|x|}$, ii. $f(x) = sin\frac{1}{x}$, iii. both $f(x) = sin\frac{1}{x}$ and $g(x) = cos\frac{1}{x}$.
- (b) Find the simplest compactification Y of $\mathbb{N}$ for which the following functions extend continuously: i. $f_p(n) = n(mod\ p)$, for a fixed p. ii. $f(n) = sin(n)$. (You can use the fact: the set partial limits of $sin(n)$ is $[-1, 1]$.)
- (c) Construct the metrizable compactifications of $(0, 1]$ in which one adds to $(0, 1]$: i. S^1, ii. Any letter of English alphabet.
- (d) Suppose the compactification X of $(0, 1]$ is metrizable. Prove that $X \setminus (0, 1]$ is connected. In particular there does not exist metrizable compactification where one adds $1 < n < \infty$ points to $(0, 1]$.

(4) Let X completely regular, non-compact and $\beta(X)$ its Stone-Čech compactification.
- (a) (the maximality of $\beta(X)$) Let Y be an arbitrary compactification of X. Prove that there exists a surjective continuous map $\beta(X) \to Y$ which is the identity on X.
- (b) Show that the cardinality of $\beta(\mathbb{N})$ is at least c^c, where $c = |[0, 1]|$.
- (c) Show that X is connected iff $\beta(X)$ is connected.
- (d) Suppose X is a discrete space.
  - (i) Show that for any $A \subset X$ the closures $\overline{A}$, $\overline{X \setminus A}$ (inside $\beta(X)$) are disjoint.
  - (ii) Show that if $U \subset \beta(X)$ is open then $\overline{U} \subset \beta(X)$ is open.
  - (iii) Show that $\beta(X)$ is totally disconnected.
- (e) Suppose X is metrizable and $y \in \beta(X) \setminus X$. Prove that y is not the limit of a sequence of points in X. In particular if X is non-compact then $\beta(X)$ is not metrizable.

(5)
- (a) Suppose (Y, d) is compact. Prove that $X \subseteq Y$ is complete iff it is compact iff it is closed.
- (b) Suppose that for some $\epsilon > 0$ every ϵ-ball in the space (X, d) has compact closure. Prove that X is complete. Does this hold if for any $x \in X$ exists $\epsilon > 0$ such that $\overline{Ball_\epsilon(x)}$ is compact?
- (c) Are locally closed/open subspaces of a complete space complete?
- (d) Prove that an infinite complete space with no isolated points is uncountable.
- (e) Prove that (X, d) is complete iff $(X, \bar{d})$ is complete for $\bar{d} = min(d, 1)$.
- (f) Suppose the spaces $\{(X_i, d_i)\}_{i=1,\dots\infty}$ are complete. Is the space $(\prod_{i=1}^{\infty} X_i, d)$ complete for $d(\{x_i\}, \{y_i\}) = sup\{\frac{\bar{d}_i(x_i, y_i)}{i}\}$?