Partial solutions of the first midterm, Algebraic Structures

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(1) Suppose $gcd(m,n)=1$, we prove that the groups are isomorphic.

Take the canonical projections, $\mathbb{Z}/mn\mathbb{Z} \overset{\pi_m}{\to} \mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/mn\mathbb{Z} \overset{\pi_n}{\to} \mathbb{Z}/n\mathbb{Z}$. Define the map $\mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, by $g \to (\pi_m(g), \pi_n(g))$. This is a homomorphism of groups, e.g. because π_m, π_n are homomorphisms.

It is injective. (If $(\pi_m(g), \pi_n(g)) = (0,0)$ then $g \in m\mathbb{Z}/mn\mathbb{Z}$ and $g \in n\mathbb{Z}/mn\mathbb{Z}$, thus $g = 0 \in \mathbb{Z}/mn\mathbb{Z}$. The map is surjective, e.g. by comparing the cardinality of the sets. Thus this map is isomorphism.

Suppose $gcd(m,n) > 1$, then the groups are not isomorphic. Indeed, $\mathbb{Z}/mn\mathbb{Z}$ has an element of order mn, while any element $g \in \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ satisfies: $\frac{mn}{gcd(m,n)} \cdot g = 0$.

(2) (a) Take a non-normal subgroup $N < G$, suppose it is contained in some normal subgroup, $N < N_1 \lhd G$. Take the canonical projection, $G \overset{\pi}{\to} G/N_1$, then $\pi(N) = \{e\} \lhd G/N_1$.
On the other hand, if $N \lhd G$ then $\pi(N) \lhd \pi(G)$, by a theorem of homomorphisms.

(b) A counterexample: $|S_3|$ is divisible by 2, but S_3 contains no normal subgroup of order 2.

(3) (a) By its definition $O(1,1)$ is a subset of $GL(2,\mathbb{R})$. To show that it is a subgroup we should check it is closed under the group operations. (skipped)
This subgroup is not normal, if $AEA^t = E$, then $g^{-1}AgE(g^{-1}Ag)^t = E$ would imply $A(gEg^t)A^t = (gEg^t)$, and the later does not necessarily hold.
A particular example: let $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{R})$, then $gEg^t = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$. And we get a contradiction, e.g. for $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(b) By its definition, $A \in SO(1,1)$ iff $A \in O(1,1)$ and in addition $det(A) = 1$. And for any $g \in O(1,1)$ holds: $det(g^{-1}Ag) = det(A)$. Thus $O(1,1)$ itself is the normalizer of $SO(1,1)$.

(c) There are just two equivalence classes of $SO(1,1)$ in $O(1,1)$: $O(1,1) = SO(1,1) \coprod (E \cdot SO(1,1))$. Therefore the quotient group is of order two. Thus $O(1,1)/SO(1,1) \approx \mathbb{Z}/2\mathbb{Z}$.

(4) As both subgroups are normal, we have $N_1 \cdot N_2 = N_2 \cdot N_1$, hence $\langle N_1, N_2 \rangle = N_1 \cdot N_2$. Furthermore, $N_1 \cap N_2 = \{e\}$, because this subgroup must divide the $gcd(n_1, n_2) = 1$. And because of this we have: $|N_1 \cdot N_2| = |N_1| \cdot |N_2|$.