Partial solutions of the second midterm, Algebraic Structures

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(1) (a) Suppose $f(x) \cdot g(x) \in I$, then $f(0)g(0) = 0$, thus $f(0) = 0$ or $g(0) = 0$, thus one of f, g belongs to I. Thus I is a prime ideal.
(b) Suppose $f(x) \notin I$, then $f(0) \neq 0$. Note that $f(x) - f(0) \in I$. Therefore $(f(x)) + I = (f(0)) + I = R$. Thus I is maximal.
(c) Suppose $I = (f(x))$, here $f(x)$ is continuous and $f(0) = 0$. Moreover, $f(x)$ does not vanish at any other point of $(-1, 1)$. (Otherwise, e.g. $x \in I$ is not divisible by $f(x)$.) Then the function $\frac{f(x)}{\sqrt{|f(x)|}}$ extends continuously to a function $g(x) \in C^0(-1, 1)$, and $g(x)$ vanishes at 0. Thus $g(x) \in I$, but $g(x)$ is not divisible by $f(x)$. Therefore I cannot be generated by just one element.

(2) Note that $36 = 4 \cdot 9$. Thus, by the classification theorem of finite abelian groups, $G \approx G_4 \times G_9$, with $|G_i| = i$. Furthermore, by this same theorem, we have:
(a) either $G_4 \approx \mathbb{Z}/_{2\mathbb{Z}} \times \mathbb{Z}/_{2\mathbb{Z}}$ or $G_4 \approx \mathbb{Z}/_{4\mathbb{Z}}$,
(b) either $G_9 \approx \mathbb{Z}/_{3\mathbb{Z}} \times \mathbb{Z}/_{3\mathbb{Z}}$ or $G_4 \approx \mathbb{Z}/_{9\mathbb{Z}}$.
Thus G is one of: $\mathbb{Z}/_{36\mathbb{Z}}$, $\mathbb{Z}/_{18\mathbb{Z}} \times \mathbb{Z}/_{2\mathbb{Z}}$, $\mathbb{Z}/_{12\mathbb{Z}} \times \mathbb{Z}/_{3\mathbb{Z}}$, $\mathbb{Z}/_{6\mathbb{Z}} \times \mathbb{Z}/_{6\mathbb{Z}}$.
These later groups are pairwise non-isomorphic, this can be checked e.g. by comparing the orders of elements.

(3) (Solution 1.) As was proved in the class: any maximal ideal is prime.
Let $a \neq 0$ be nilpotent. Fix n such that $a^n = 0$ but $a^{n-1} \neq 0$. Then $a^{n-1} \cdot a = 0 \in I$. Thus $a^{n-1} \in I$. Hence $a \in I$.
(Solution 2.) As was proved in the class: if $I \subset R$ is a maximal ideal then $R/_I$ is a field.
For any nilpotent element $a \in R$ take its image $[a] \in R/_I$. The image is nilpotent as well, in a field, thus $[a] = 0$. Thus $a \in I$.

(4) (a) Suppose $K < G$ is of order p then the order of $\phi(K) \approx K/_{ker(\phi) \cap K}$ divides p. Thus, either $\phi(K) = \{e\}$ or $|\phi(K)| = p$.
(b) Suppose $|G| = p^n m$, where $gcd(p, m) = 1$. Let $K \in Syl_p(G)$, i.e. $|K| = p^n$. Note that $ker(\phi) < G$, thus $|ker(\phi)| = p^{\tilde{n}}\tilde{m}$, for some $\tilde{n} \leq n$ and $\tilde{m} \mid m$. Thus $|H| = |G/_{ker(\phi)}| = p^{n-\tilde{n}}\frac{m}{\tilde{m}}$.
We have: $\phi(K) \approx \frac{K}{K \cap ker(\phi)}$. As $\Big(K \cap ker(\phi)\Big) \leq K$ one has: $|K \cap ker(\phi)| = p^{n'}$, for some $n' \leq \tilde{n}$. Thus $|\phi(K)| = p^{n-n'}$. But $\phi(K) \leq H$, thus $n - n' \leq n - \tilde{n}$, i.e. $n' \geq \tilde{n}$. Therefore $n' = \tilde{n}$. Hence $\phi(K) \in Syl_p(H)$.

(5) (a) If R is not a unital ring then $R^\times = \varnothing$ and the statement is trivial. Thus we assume $1 \in R$.
First we prove: $\phi(1) = 1$. Indeed, present any element of R in the form $\phi(a)$, for some a. Then $\phi(1)\phi(a) = \phi(1 \cdot a) = \phi(a) = \phi(a)\phi(1)$. Thus, by the uniqueness of the unit element, $\phi(1) = 1$.
Now, if $a \in R^\times$ then $a^{-1} \in R$ and thus $1 = \phi(a \cdot a^{-1}) = \phi(a)\phi(a^{-1})$, i.e. $\phi(a)^{-1} = \phi(a^{-1})$. Hence $\phi(R^\times) = R^\times$.
(b) Note that $\phi(1) = 1$, thus $\phi(n) = n$ for any $n \in \mathbb{Z}$. Thus ϕ acts as identity on $\mathbb{Q}$.
(c) As a vector space over $\mathbb{Q}$ the ring is: $\mathbb{Q} < 1 > +\mathbb{Q} < x >$. Thus, for any $\phi \in Aut(R)$ it is enough to check its action on $\mathbb{Q} < 1 >$ and $\mathbb{Q} < x >$.
As was shown in (b), ϕ acts as identity on $\mathbb{Q} < 1 >$.
Note that $x^2 = 0 \in R$, therefore $\phi(x) \in (x)$, i.e. $\phi(x) = c \cdot x$ for some $c \in \mathbb{Q}$. Therefore any automorphism of R acts by $a \cdot 1 + b \cdot x \overset{\phi}{\to} a \cdot 1 + bc \cdot x$.
It remains to observe that any such map is indeed an automorphism of R. It is a $\mathbb{Q}$-linear map, by construction. And it is multiplicative, by direct check. Finally, it is invertible iff $c \neq 0$. Thus, as a set, $Aut(R) = \mathbb{Q} \setminus \{0\}$. Finally, we check the group structure, $\phi_{c_1} \circ \phi_{c_2} = \phi_{c_1 \cdot c_2}$. Therefore $Aut(R) = \mathbb{Q} \setminus \{0\}$ as a group.