Partial solutions of Moed.A, Algebraic Structures

(201.1.7031) 08.02.2018 Ben Gurion University

(1) (a) A proof: consider the map $GL(n,\Bbbk) \stackrel{det}{\to} \Bbbk^\times$. This is a homomorphism of groups. This map is surjective, because any $\lambda \in \Bbbk$ is achieved e.g. as $det(\lambda 1\!\!1_{1\times 1} \oplus 1\!\!1_{(n-1)\times(n-1)})$. And the kernel of this map is precisely $SL(n,\Bbbk)$. Thus $SL(n,\Bbbk) \lhd GL(n,k)$ and ${}^{GL(n,\Bbbk)}/_{SL(n,\Bbbk)} \approx \Bbbk^\times$.

(b) A counterexample: take the group of quaternions, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. One has: $Z(Q_8) = \{\pm 1\}$ and ${}^{Q_8}/_{Z(Q_8)} \approx {}^{\mathbb{Z}}/_{2\mathbb{Z}} \oplus {}^{\mathbb{Z}}/_{2\mathbb{Z}}$, i.e. abelian but non-trivial.

(c) As $[G,G]$ is generated by the elements of the form $aba^{-1}b^{-1}$, it is enough to check that $\phi(aba^{-1}b^{-1}) = 1\!\!1_H$. And this is immediate by the properties of homomorphisms.
A counterexample to the second statement: let $G = S_3 = \{1, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$ and take $H = (1,\tau)$, with its embedding. Note that $H < S_3$ is not a normal subgroup.

(2) As $51 = 3 \cdot 17$ we have by Sylow's theorem (3): the number of 17-Sylow subgroups of G is of the form $1 + 17\mathbb{Z}$ and divides 3. Thus there is only one 17-Sylow subgroup, hence it is normal. In the same way, $Syl_3(G)$ consists of just one element, hence ${}^{\mathbb{Z}}/_{3\mathbb{Z}} \lhd G$. Thus ${}^{\mathbb{Z}}/_{3\mathbb{Z}}$ and ${}^{\mathbb{Z}}/_{17\mathbb{Z}}$ are normal subgroups of G

These subgroups are mutually disjoint (as $gcd(3,17) = 1$), and $|{}^{\mathbb{Z}}/_{3\mathbb{Z}}| \cdot |{}^{\mathbb{Z}}/_{17\mathbb{Z}}| = |G|$. Thus, as has been proved in homeworks, $G \approx {}^{\mathbb{Z}}/_{3\mathbb{Z}} \times {}^{\mathbb{Z}}/_{17\mathbb{Z}} \approx {}^{\mathbb{Z}}/_{51\mathbb{Z}}$.

(3) (a) Define $N(a+ib) = a^2 + b^2$. Note that $N(5+7i) = N(7+5i) = 2 \cdot 37$. Therefore any common divisor c of the two elements satisfies: $N(c) = 2$ or $N(c) = 37$.
In the first case we get: $c = \pm 1 \pm i$. The second case is realized only by $\pm 1 \pm 6i$, $\pm 6 \pm i$.
By direct check, neither of $\pm 1 \pm 6i$, $\pm 6 \pm i$ divides neither of $5+7i$, $7+5i$. And the numbers $\pm 1 \pm i$ divide $5+7i$, $7+5i$. (Note that $\pm 1 \pm i$ differ by invertibles.) Therefore $gcd(5+7i, 7+5i)$ is either of $\pm 1 \pm i$.

(b) (R/I is an integral domain) $\iff \Big(\forall a,b \in R,\ (a+I)(b+I) = I$ implies that $a+I = I$ or $b+I = I\Big) \iff$ ($\forall a,b \in R$, $ab \in I$ implies $a \in I$ or $b \in I$) $\iff$ (I is a prime ideal in R).

(c) Suppose $J \subset R$ is a prime ideal that contains I. Then $J \ni x^4 = x \cdot x^3$, hence $J \ni x$. Also note that $I \ni y^4$, hence $J \ni y$. Thus either $J = (x,y)$, or $J = R$.

(4) (a) Note that $(y^3 - x^4) \cdot M = \{0\}$, and $y^3 - x^4 \in R$ is not a zero divisor. Thus the whole M is torsion.
Note that M is a cyclic module, denote its generator by z. Thus any homomorphism $M \stackrel{\phi}{\to} R$ is determined by the image of the generator, $\phi(z)$. Note that $(y^3 - x^4)\phi(z) = \phi((y^3 - x^4)z) = \phi(0) = 0 \in R$, i.e. $\phi(z)$ is a zero divisor in R. As R is a integral domain, $\phi(z) = 0$. Thus $Hom(M,R) = \{0\}$.

(b) *Solution 1.* Recall that $\Bbbk[x]$ is a PID, thus A can be brought to the Smith normal form by $A \to UAV$ for some $U \in GL(m,R)$, $V \in GL(n,R)$.
Thus the given system is equivalent to a system with a diagonal matrix, i.e the equations are $\{a_i x_i = 0\}$.
Now, as R is an integral domain, if $a_i \neq 0$ then $x_i = 0$, otherwise x_i is arbitrary. Thus the set of solutions is a free submodule of $R^{\oplus n}$.
Solution 2. The set of solutions of the system $A \cdot \underline{x} = 0$ is a submodule of the module R^n. The later is a free module over R. Then, as R is a PID, the set of solution is itself a free module.