Algebraic Structures- Solutions of Homework 4

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1 Question 1.

1.1 (a):

(i) $G_1 \times \{e\} \lhd G_1 \times G_2$**:** For every $g_1, h_1 \in G_1, g_2 \in G_2$,

$$(g_1, g_2)(h_1, e)(g_1, g_2)^{-1} = (g_1 h_1 g_1^{-1}, e) \in G_1 \times \{e\}.$$

The isomorphism is given by-

$$\phi : G_1 \times G_2 / G_1 \times \{e\} \to G_2, \quad \phi((g_1, g_2)(G_1 \times \{e\})) = g_2.$$

(ii) $\mathbb{Z} \lhd \mathbb{R}$**:** For every $r \in \mathbb{R}, n \in \mathbb{Z}$, $r + n + (-r) = n \in \mathbb{Z}$. The isomorphism is given by-

$$\phi : \mathbb{R}/\mathbb{Z} \to \mathcal{S}^1, \quad \phi(r + \mathbb{Z}) = r - [r],$$

where $[r]$ is the greatest integer that is smaller than r.

(iii) $\mathbb{Z}^2 \lhd \mathbb{R}^2$ **:** For every $(r_1, r_2) \in \mathbb{R}^2$ and $(n_1, n_2) \in \mathbb{Z}^2$,

$$(r_1, r_2) + (n_1, n_2) + (-r_1, -r_2) = (n_1, n_2) \in \mathbb{Z}^2.$$

The isomorphism given by-

$$\phi : \mathbb{R}^2/\mathbb{Z}^2 \to \mathcal{S}^1 \times \mathcal{S}^1, \quad \phi((r_1, r_2)\mathbb{Z}^2) = (r_1 - [r_1], r_2 - [r_2]).$$

(iv) $SL_n(\mathbb{K}) \lhd GL_n(\mathbb{K})$ **:** For every $A \in GL_n(\mathbb{K})$ and $B \in SL_n(\mathbb{K})$,

$$\det(A^{-1}BA) = \det(A^{-1}) \det(B) \det(A) = \det(B) = 1$$

so $A^{-1}BA \in SL_n(\mathbb{K})$. The isomorphism is given by-

$$\phi : GL_n(\mathbb{K})/SL_n(\mathbb{K}) \to \mathbb{K}^{\times}, \quad \phi(A \cdot SL_n(\mathbb{K})) = \det(A).$$

(v) $SO_n(\mathbb{K}) \lhd O_n(\mathbb{K})$ **:** For every $A \in O_n(\mathbb{K})$ and $B \in SO_n(\mathbb{K})$,

$$\det(A^{-1}BA) = \det(B) = 1 \Rightarrow A^{-1}BA \in SO_n(\mathbb{K}).$$

The isomorphism is given by-

$$\phi : O_n(\mathbb{R})/SO_n(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}, \quad \phi(A \cdot SO_n(\mathbb{R})) = \det(A).$$

(vi) $\mathbb{Z}/n\mathbb{Z} \lhd D_{2n}$: The group D_{2n} is generated by two elements r, s, where r is a rotation of order n, s is a reflection of order 2 and $r^{-1} = srs$. Then $\langle r \rangle \leq D_{2n}$ is a subgroup of order n and hence of index 2, therefore $\langle r \rangle \lhd D_{2n}$ and $\langle r \rangle \approx \mathbb{Z}/n\mathbb{Z}$. The isomorphism is given by

$$\phi : D_{2n}/\langle r \rangle \to \mathbb{Z}/2\mathbb{Z}, \quad \phi(x \cdot \langle r \rangle) = k - 1(\mod 2)$$

where $x = r^{n_1} s r^{n_2} s \cdot ... \cdot s r^{n_k} \in D_{2n}$, as $x \cdot \langle r \rangle = s^{k-1} \cdot \langle r \rangle$.

(vii) $N = \{(1), (12)(34), (13)(24), (14)(23)\} \lhd S_4$: For every $\sigma \in S_4$ we know that $\sigma^{-1}(ij)(kl)\sigma = (\sigma(i), \sigma(j))(\sigma(k), \sigma(l)) \in N$. The isomorphism is given by

$$\phi : S_4/N = \{N, (12) \cdot N, (13) \cdot N, (14) \cdot N, (123) \cdot N, (132) \cdot N\} \to S_3,$$
$$\phi(N) = (1), \phi((12) \cdot N) = (12), \phi((13) \cdot N) = (13), \phi((14) \cdot N) = (23) \cdot N,$$
$$\phi((123) \cdot N) = (123), \phi((132) \cdot N) = (132).$$

1.2 (b):

- If H is a subgroup of G with $|G| : |H| = 2$, then the index of H in G is equal to 2, which means that there are exactly 2 right co-sets of H and 2 left co-sets, i.e., there exists $a \in G$ such that $a \notin H$ and $H \cup aH = G$. It follows that $Ha \neq H$ and therefore the 2 right co-sets of H must be H and Ha, so
$$G = H \cup aH = H \cup Ha.$$
Therefore, $Ha = G \setminus H$ and $Ha = G \setminus aH$, so $aH = Ha$. As H has only 2 right/left co-sets, we basically showed that H is a normal subgroup of G.

- Moreover, it is easily seen that
$$G/H = \{H, aH\}$$
is a group of order 2 and hence isomorphic to the group $\mathbb{Z}/2\mathbb{Z}$.

1.3 (c):

The answer is yes! We know that $\mathbb{R}^2 \setminus \{(0,0) \approx \mathbb{C}^\times$ and $SO_2(\mathbb{R}) \approx \mathcal{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, therefore the action of $SO_2(\mathbb{R})$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ is the same as the action of $\mathcal{S}^1$ on $\mathbb{C}^\times$ and we have $\mathbb{C}^\times/\mathcal{S}^1 \approx \mathbb{R}_{>0}$.

2 Question 2.

2.1 (a):

- S_3 is not simple, as $H = \langle (123) \rangle$, which is the subgroup of S_2 generated by a cycle of length 3, is of order 3 and hence of index 2, so $H \lhd S_3$.

- S_4 is not simple as A_4, which is the subgroup of S_4 consists of all even permutations, is a normal subgroup of S_4.

- D_{2n} is never simple, as it always contain a subgroup of index 2, that is the subgroup of D_{2n} generated by the rotation, and hence is a normal subgroup of D_{2n}.

2.2 (b):

Let G be a group of order 62. By Cauchy's theorem, there exists $g \in G$ of order 31, let $H := \langle g \rangle$, then H is a subgroup of G of order 31 and hence the index of H in G is equal to 2. From part (b) in question 1 we know that $H \lhd G$ and hence G is not simple.

3 Question 3.

3.1 (c):

Let $N \lhd G$ and suppose G/N is abelian. Then, for every $g_1, g_2 \in G$,

$$(g_1N)(g_2N) = (g_2N)(g_1N) \Longrightarrow (g_1g_2)N = (g_2g_1)N \Longrightarrow (g_1^{-1}g_2^{-1}g_1g_2)N = N$$

which means that $g_1^{-1}g_2^{-1}g_1g_2 = [g_1, g_2] \in N$ for all $g_1, g_2 \in G$ and therefore the group generated by all commutators is a subset of N, i.e., $[G, G] \leq N$.

3.2 (d):

Let $H \leq G$ and suppose that $[G, G] \leq H$.Then, for every $g \in G, h \in H$,

$$g^{-1}hg = hh^{-1}g^{-1}hg = h[h, g] \in H \cdot [G, G] \subset H \cdot H = H,$$

i.e., $g^{-1}hg \in H$ and $H \lhd G$.

4 Question 5.

4.1 (a):

Assume that $N_1, N_2 \lhd G$ and $N_1 \cap N_2 = \{e\}$. For every $g_1 \in G_1$ and $g_2 \in G_2$, we have

$$g_2^{-1}g_1^{-1}g_2g_1 = g_2^{-1}(g_1^{-1}g_2g_1) \in G_2 \cdot (g_1^{-1}G_2g_1) = G_2 \cdot G_2 = G_2$$

and similarly

$$g_2^{-1}g_1^{-1}g_2g_1 = (g_2^{-1}g_1^{-1}g_2)g_1 \in (g_2^{-1}G_1g_2) \cdot G_1 = G_1 \cdot G_1 = G_1,$$

then

$$g_2^{-1}g_1^{-1}g_2g_1 \in G_1 \cap G_2 = \{e\} \Rightarrow g_2^{-1}g_1^{-1}g_2g_1 = e \Rightarrow g_1g_2 = g_2g_1.$$

5 Question 7.

For every $x, g \in G$ and $S \subseteq G$, recall the notations $x^g := g^{-1}xg$ and $S^g := \{x^g | x \in S\} = g^{-1}Sg$.

5.1 (a): $N_G(S) := \{g \in G | S^g = S\}$ is a subgroup of G.

- $x^e = x$ for every $x \in G$, so $S^e = S$ and hence $e \in N_G(S)$.

- If $g, h \in N_G(S)$, then $S^g = g^{-1}Sg = S$ implies $S^{g^{-1}} = gSg^{-1} = S$ and so

  $$\begin{aligned} S^{g^{-1}h} &= \{x^{g^{-1}h} | x \in S\} = \{(g^{-1}h)^{-1}x(g^{-1}h) | x \in S\} = \{h^{-1}(gxg^{-1})h | x \in S\} = h^{-1}(gSg^{-1})h \\ &= h^{-1}S^{g^{-1}}h = h^{-1}Sh = S^h = S \end{aligned}$$

  which implies that $g^{-1}h \in N_G(S)$ and hence that $N_G(S) \leq G$.

- For every $g \in N_G(S)$ and $h = x_1 \cdot ... \cdot x_m \in \langle S \rangle$, where x_i or x_i^{-1} in S, we have

  $$g^{-1}hg = (g^{-1}x_1g) \cdot ... \cdot (g^{-1}x_mg)$$

  and that is a product of elements from $g^{-1}Sg = S$ and inverses of such, therefore $g^{-1}hg \in \langle S \rangle$. Therefore $\langle S \rangle \lhd N_G(S)$.

- Let H be a subgroup of G with the property that $\langle S \rangle \lhd H$. For every $h \in H$, we have $h^{-1}\langle S \rangle h = \langle S \rangle$ and it is easily seen that $h^{-1}\langle S \rangle h = \langle h^{-1}Sh \rangle$, so $\langle S \rangle = \langle h^{-1}Sh \rangle$ and hence $S = h^{-1}Sh$. That means that $h \in N_G(S)$ and hence $H \subseteq N_G(S)$ and $N_G(S)$ is the greatest subgroup of G for which $\langle S \rangle$ is a normal subgroup of it.

5.2 (b): $C_S(G) := \{g \in G | \forall s \in S, s^g = s\}$ is a subgroup of G.

- $s^e = s$ fr all $s \in S$ and hence $e \in C_G(S)$.

- If $g, h \in C_G(S)$, then $s = gsg^{-1}$ and

  $$s^{g^{-1}h} = (g^{-1}h)^{-1}s(g^{-1}h) = h^{-1}(gsg^{-1})h = h^{-1}sh = s$$

  which implies that $g^{-1}h \in C_G(S)$ and then $C_G(S) \leq G$.

- It is not always true that $C_G(g^{-1}Sg) = g^{-1}C_G(S)g$.

5.3 (c): $Z(G) := \{g \in G | \forall h \in G, hg = gh\} \lhd G$.

- $eh = he = h$ for all $h \in G$ and hence $e \in Z(G)$.

- If $g_1, g_2 \in Z(G)$, then for all $h \in G$,

  $$h(g_1g_2) = (hg_1)g_2 = (g_1h)g_2 = g_1(hg_2) = g_1(g_2h) = (g_1g_2)h$$

  which eans that $g_1g_2 \in Z(G)$ and that $Z(G) \leq G$.

- If $x \in G$ and $g \in Z(G)$, then $x^{-1}gx = g \in Z(G)$ and hence $Z(G) \lhd G$.

- By its definition

  $$C_G(G) = \{g \in G | \forall h \in G, h^g = h\} = \{g \in G | \forall h \in G, hg = gh\} = Z(G).$$

5.4 (d):

$Z(GL_n(\mathbb{K}))$ is the subgroup of $GL_n(\mathbb{K})$ consists of all $n \times n$ matrices over $\mathbb{K}$ which are invertible and commute with every other $n \times n$ matrix: we know that this set of all matrices is exactly the set of all scalar matrices over $\mathbb{K}$.

5.5 (e):

Suppose that $G/Z(G)$ is cyclic, so there exists $a \in G$ for which

$$G/Z(G) = \langle aZ(G) \rangle.$$

Therefore, for every $g_1, g_2 \in G$, there exist $n_1, n_2 \in \mathbb{Z}$ for which

$$g_1 Z(G) = a^{n_1} Z(G) \quad \text{and} \quad g_2 Z(G) = a^{n_2} Z(G),$$

which imply that $g_1 = a^{n_1} z_1, g_2 = a^{n_2} z_2$ for some $z_1, z_2 \in Z(G)$ and that

$$g_1 g_2 = a^{n_1} z_1 a^{n_2} z_2 = a^{n_1} a^{n_2} z_1 z_2 = a^{n_2} a^{n_1} z_2 z_1 = a^{n_2} z_2 a^{n_1} z_1 = g_2 g_1,$$

i.e., G is abelian.