Algebraic Structures- Solutions of Homework 5

written by Motke Porat

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1 Question 1.

1.1 (c):

Assume $\phi : G \to H$ is a group homomorphism and let $g \in G$ be such that $n := ord_G(g) < \infty$. Then it follows that

$$(\phi(g))^n = \phi(g^n) = \phi(e_G) = e_H$$

and hence we must have that $ord_H(\phi(g))|n$, i.e.,

$$ord_H(\phi(g))|ord_G(g).$$

1.2 (e):

Assume $N \lhd G$ and let $g \in G$ be of a finite order $n = ord_G(g) < \infty$. Then

$$(gN)^n = g^n N = e_G N = N = e_{G/N}$$

and hence $ord_{G/N}(gN)|n$, i.e.,

$$ord_{G/N}(gN)|ord_G(g).$$

2 Question 3.

2.1 (c):

Assume $\phi : G \to H$ is a group homomorphism and let $A, B \leq G$.

- If $g \in \phi(\langle A, B\rangle)$ then $g = \phi(x)$ where $x = x_1 \cdot ... \cdot x_m$ and $x_1, ..., x_m \in A \cup B$, and hence

  $$g = \phi(x_1 \cdot ... \cdot x_m) = \phi(x_1) \cdot ... \cdot \phi(x_m) \in \langle \phi(A), \phi(B)\rangle$$

  as $\phi(x_i) \in \phi(A) \cup \phi(B)$ for all $i = 1, ..., m$, i.e.,

  $$\phi(\langle A, B\rangle) \subseteq \langle \phi(A), \phi(B)\rangle.$$

- If $g \in \langle \phi(A), \phi(B)\rangle$ then $g = x_1 \cdot ... \cdot x_m$ where $x_1, ..., x_m \in \phi(A) \cup \phi(B)$ and then $x_i = \phi(y_i)$ for $y_1, ..., y_m \in A \cup B$, which imply that

  $$g = \phi(y_1) \cdot ... \cdot \phi(y_m) = \phi(y_1 \cdot ... \cdot y_m) \in \phi(\langle A, B\rangle),$$

  i.e., $\langle \phi(A), \phi(B)\rangle \subseteq \phi(\langle A, B\rangle)$.

- It is not always true: let $G = \mathbb{Z}_2 \times \mathbb{Z}_2, H = \mathbb{Z}_2$ and the subgroups

$$A = \{(0,0),(1,0)\}, B = \{(0,0),(1,1)\}.$$

Consider $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ given by $\phi(a,b) = a$. So-

$$\phi(A) = \{0,1\}, \phi(B) = \{0,1\} \Longrightarrow \phi(A) \cap \phi(B) = \{0,1\}$$

but $\phi(A \cap B) = \phi(\{(0,0)\}) = \{0\}$.

2.2 (d):

- Assume that $N \lhd \phi(G)$ and recall that

$$\phi^{-1}(N) = \{g \in G : \phi(g) \in N\}$$

is a subgroup of G. This is a normal subgroup of G, as if $h \in \phi^{-1}(N)$ and $g \in G$, then

$$\phi(g^{-1}hg) = \phi(g^{-1})\phi(h)\phi(g) = \phi(g)^{-1}\phi(h)\phi(g) \in N$$

as $\phi(h) \in N, \phi(g) \in \phi(G)$ and $N \lhd \phi(G)$.

- Assume that $N \subset \phi(G)$ and $\phi^{-1}(N) \lhd G$. For every $n \in N$ and $x \in \phi(G)$, there exists $g \in G$ and $h \in \phi^{-1}(N)$ such that $x = \phi(g)$ and $n = \phi(h)$, so

$$x^{-1}nx = \phi(g)^{-1}\phi(h)\phi(g) = \phi(g^{-1}hg) \in N$$

as $g^{-1}hg \in \phi^{-1}(N)$.

- It is not always true: let G be any group which contains a non-normal subgroup K and let $\phi : G \to H$ be $\phi \equiv 1_H$. Clearly ϕ is an homomorphism from G to H, $\phi(K) = \phi(G) = \{1_G\}$ and $K$is not a normal subgroup of G.

3 Question 4.

3.1 (b):

- Consider the function

$$\phi : Q \to \mathcal{S}^1, \quad \phi(q) = e^{2\pi qi} = \cos(2\pi q) + i\sin(2\pi q).$$

Clearly ϕ is an homomorphism, as

$$\phi(q_1 + q_2) = e^{2\pi(q_1+q_2)i} = e^{2\pi q_1 i}e^{2\pi q_2 i} = \phi(q_1)\phi(q_2)$$

and if $q = m/n$ where $m, n \in \mathbb{Z}$, then

$$\phi(q)^n = \phi(nq) = \phi(m) = e^{2\pi mi} = 1$$

so $\phi(Q) = Tor(\mathcal{S}^1)$. Finally, $q \in \ker\phi \leftrightarrow e^{2\pi qi} = 1 \leftrightarrow q \in \mathbb{Z}$ which means that $\ker\phi = \mathbb{Z}$, and from the homomorphism theorem we get that

$$Q/\mathbb{Z} \approx Tor(\mathcal{S}^1).$$

- We know that $\mathbb{R}/\mathbb{Z} \approx \mathcal{S}^1$ and $Q/\mathbb{Z} \approx Tor(\mathcal{S}^1)$, therefore

$$\mathcal{S}^1/Tor(\mathcal{S}^1) \approx (\mathbb{R}/\mathbb{Z})/(Q/\mathbb{Z})$$

while from the third homomorphism theorem we know that

$$(\mathbb{R}/\mathbb{Z})/(Q/\mathbb{Z}) \approx \mathbb{R}/Q$$

and therefore

$$\mathbb{R}/Q \approx \mathcal{S}^1/Tor(\mathcal{S}^1).$$

3.2 (c): No! here is an example:

Let $G = GL_2(\mathbb{R})$, which is clearly not abelian, and let the matrices $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in G. It is easy to check that $A^2 = I_2$ and $B^2 = I_2$, so $A, B \in Tor(G)$. However, $AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \notin Tor(G)$, as $(AB)^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \neq I_2$.

4 Question 5.

4.1 (b):

$$\begin{aligned}
(g_1, g_2) \in N_{G\times G}(\phi(G)) &\iff (g_1, g_2)^{-1}\phi(G)(g_1, g_2) = \phi(G) \\
&\iff \forall g \in G, (g_1, g_2)^{-1}(g, g)(g_1, g_2) \in \phi(G) \\
&\iff \forall g \in G, g_1^{-1} g g_1 = g_2^{-1} g g_2 \\
&\iff \forall g \in G, g g_1 g_2^{-1} = g_1 g_2^{-1} g \\
&\iff g_1 g_2^{-1} \in Z(G)
\end{aligned}$$

so

$$N_{G\times G}(\phi(G)) = \{(g_1, g_2) \quad | \quad g_1 \in Z(G)g_2, g_2 \in G\}.$$

Therefore,

$$\begin{aligned}
\phi(G) \lhd G \times G \iff N_{G\times G}(\phi(G)) = G \times G \iff \forall g_2 \in G, Z(G)g_2 = G \\
\iff Z(G) = G \iff G \text{ is abelian.}
\end{aligned}$$

5 Question 6.

5.1 (a):

Let G be a group of order $p \cdot q$, with $p < q$ primes. From Cauchy theorem, there exists $x \in G$ of order q, so let $H := \langle x \rangle$ be the subgroup of G generated by x. **This is the only subgroup of G of order q:** If $K \leq G$ is of order q and $K \neq H$, then $H \cap K \leq H$ and so $H \cap K = \{e\}$ or $H \cap K = H$, while the second can not hold as $K \neq H$, and therefore $H \cap K = \{e\}$. Then we get that

$$|H \cdot K| = |H| \cdot |K| = q^2 > pq = |G|$$

which is a contradiction and hence $K = H$.

As H is the only subgroup of G of order q, for every $g \in G$ we have $|g^{-1}Hg| = |H| = q$ so we must have $g^{-1}Hg = H$ which means that $H \lhd G$. Finally, G/H is a group of order p so it is isomorphic to $\mathbb{Z}_p$, i.e.,

$$\mathbb{Z}_q \approx H \lhd G, \mathbb{Z}_p \approx G/H.$$

5.2 (f):

Let G be a group that is not abelian and $|G| = 2p$, where $2 < p$ is prime. By Cauchy theorem, there exist $a, b \in G$ of orders $2, p$, i.e., $ord(a) = 2, ord(b) = p$. If $ab = ba$ then $(an)^n = a^n b^n$ and order of ab will be the smallest common multiple of 2 and p, which is equal to $2p$ as $p > 2$ is prime. Then G contains an element (ab) of order $2p$ and so it must be cyclic and then abelian, which is a contradiction. So, we must have $ab \neq ba$. We found two elements a and b in G, which are not commute of orders 2 and p, that is exactly how the group D_{2p} is defined and hence $G \approx D_{2p}$.

6 Question 8.

Let $\mathbb{F}_2$ be a field with 2 elements, say $\mathbb{F} = \{0, 1\}$. The group $GL_2(\mathbb{F}_2)$ acts on $\mathbb{F}_2^2$ in a trivial way and it has 2 different orbits: $\{(0,0)\}$ and $X = \mathbb{F}_2^2 \setminus \{(0,0)\}$. For every $A \in GL_2(\mathbb{F}_2)$, A acts on X just by permutations (as A is invertible), faithfully and transitively, therefore $GL_2(\mathbb{F}_2) \approx Sym(X) \approx S_3$.

7 Question 9.

7.1 (c):

This is not true: let $G = S_3, H_1 = \langle(123)\rangle$ and $H_2 = \langle(12)\rangle$. Clearly, $G = H_1 \cdot H_2$ and $|G| = 6 = 3 \cdot 2 = |H_1| \cdot |H_2|$, but $S_3 \ncong H_1 \times H_2$, as $\{(1)\} \times H_2$ is a normal subgroup of $H_1 \times H_2$, while S_3 has no normal subgroups of order 2.