Algebraic Structures- Solutions of Homework 7

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1 Question 1.

1.1 (g):

For every $\sigma \in S_n$, there exist $\tau_1, ..., \tau_k$ all cycles of lengths $\ell_1, ..., \ell_k$ for which

$$\sigma = \tau_1 \cdot ... \cdot \tau_k,$$

each $\tau_i = (a_i^1, ..., a_i^{\ell_i})$ can be written as

$$\tau_i = (a_i^1, ..., a_i^{\ell_i}) = (a_i^1, a_i^2) \cdot .... \cdot (a_i^1, a_i^{\ell_i}) = \tau_i^{(2)} \cdot ... \cdot \tau_i^{(\ell_i)}$$

and therefore

$$\sigma = \prod_{i=1}^{k} \tau_i = \prod_{i=1}^{k} \prod_{j=2}^{\ell_i} \tau_i^{(j)}$$

is a product of cycles of length 2, therefore $S_n = \langle \{(ij) : 1 \leq i, j \leq n\} \rangle$. Finally, as $(i, j) = (j, i)$ it follows that $S_n = \langle \{(i, j) : 1 \leq i < j \leq n\} \rangle$.

2 Question 2.

Let $\mathcal{B} = \{x_1, ..., x_n\}$ be a basis of $\mathbb{R}^n$, $\phi : S_n \to GL_n(\mathbb{R})$ is defined by

$$(\phi(\sigma))(x_i) = x_{\sigma(i)}, \quad \forall 1 \leq i \leq n.$$

2.1 (b):

In part (a) you showed that $\phi : S_n \to GL_n(\mathbb{R})$ is a $1-1$ homomorphism, therefore $\phi : A_n \to \phi(A_n)$ is an isomorphism and $A_n \approx \phi(A_n)$. We will now show that $\phi(A_n) = \phi(S_n) \cap SL_n(\mathbb{R})$:

- If $\sigma \in A_n$ then $\phi(\sigma)$ is the permutation matrix corresponding to the permutation σ and as σ is even (in A_n), we have

  $$\det(\phi(\sigma)) = (-1)^{sign(\sigma)} = (-1)^2 = 1,$$

  so $\phi(\sigma) \in \phi(S_n) \cap SL_n(\mathbb{R})$ and hence $\phi(A_n) \subseteq \phi(S_n) \cap SL_n(\mathbb{R})$.

- If $\varphi \in \phi(S_n) \cap SL_n(\mathbb{R})$, then $\varphi = \phi(\sigma)$ for some $\sigma \in S_n$, but as $\varphi = \phi(\sigma) \in SL_n(\mathbb{R})$ it means that $\sigma \in A_n$ (as $\det(\phi(\sigma)) = (-1)^{sign(\sigma)}$) and hence $\varphi \in \phi(A_n)$, i.e., $\phi(S_n) \cap SL_n(\mathbb{R}) \subseteq \phi(A_n)$.

Therefore, $A_n \approx \phi(S_n) \cap SL_n(\mathbb{R})$.

2.2 (e):

Let $G \leq S_n$ and assume there exist $\sigma \in G$ that is odd, i.e., that $\sigma \notin A_n$. Define $H := G \cap A_n$ then $H \subseteq G$ and it is a subgroup as the intersection of 2 subgroups of S_n, so $H \leq G$. As $\sigma \notin A_n$, it follows that $\sigma \cdot A_n = S_n \setminus A_n$ and then

$$H \cup \sigma \cdot H = (G \cap A_n) \cup (\sigma \cdot G \cap \sigma \cdot A_n) = (G \cap A_n) \cup (G \cap (S_n \setminus A_n)) = G,$$

which means that $\{a \cdot H \mid a \in G\} = \{H, \sigma \cdot H\}$ and hence the index of H in G is exactly 2 and H is a normal subgroup of G.

3 Question 3.

$\mathbb{Z}/n\mathbb{Z}$: The group is abelian and so every subgroup is normal, therefore let $n = p_1^{n_1} \cdot ... \cdot p_k^{n_k}$ where $p_1, ..., p_k$ are distinct prime integers and we can take

$$\{1\} \lhd \langle p_1^{n_1-1} p_2^{n_2} ... p_k^{n_k} \rangle \lhd ... \lhd \langle p_1 p_2^{n_2} ... p_k^{n_k} \rangle \lhd \langle p_2^{n_2} ... p_k^{n_k} \rangle \lhd ... \langle p_k^{n_k} \rangle \lhd ... \lhd \langle p_k \rangle \lhd \langle 1 \rangle = \mathbb{Z}_n,$$

where it is easily seen that

$$ord(p_1^{m_1} ... p_k^{m_k}) = p_1^{n_1-m_1} \cdot ... \cdot p_k^{n_k-m_k}, \quad \text{for} \quad m_1 \leq n_1, ..., m_k \leq n_k,$$

so it follows that all the quotient groups are of prime order and so simple.

D_{2n}: As $\sigma \in D_{2n}$ is of order n we have the series

$$\{e\} \lhd \langle \sigma \rangle \lhd D_{2n}$$

but $\langle \sigma \rangle \approx \mathbb{Z}_n$ is not necessary simple, so we have to use the composition series we built above for $\mathbb{Z}_n$, to get (if $n = p_1^{n_1} \cdot ... \cdot p_k^{n_k}$)

$$\{e\} \lhd \langle \sigma^{p_1^{n_1-1} p_2^{n_2} ... p_k^{n_k}} \rangle \lhd ... \lhd \langle \sigma^{p_1 p_2^{n_2} ... p_k^{n_k}} \rangle \lhd \langle \sigma^{p_2^{n_2} ... p_k^{n_k}} \rangle \lhd ... \langle \sigma^{p_k^{n_k}} \rangle \lhd ... \lhd \langle \sigma \rangle \lhd D_{2n},$$

and this is a composition series for D_{2n}.

Q_8: Notice that

$$\{1\} \leq \langle -1 \rangle = \{-1, 1\} \leq \langle \mathbf{i} \rangle = \{1, -1, \mathbf{i}, \mathbf{-i}\} \leq Q_8$$

and that

$$|\langle -1 \rangle| = |\langle \mathbf{i} \rangle| / |\langle -1 \rangle| = |Q_8| / |\langle \mathbf{i} \rangle| = 2$$

implies (as a subgroup of index 2 must be normal subgroup) that

$$\{1\} \lhd \langle -1 \rangle \lhd \langle \mathbf{i} \rangle \lhd Q_8$$

and this is a composition series, as the factors are all of prime order and so simple.

S_n: If $n \neq 4$, then we have the composition series

$$\{(1)\} \lhd A_n \lhd S_n,$$

as we know that $A_n/\{(1)\} \approx A_n$ and $S_n/A_n \approx \mathbb{Z}_2$ are simple. If $n = 4$, we have the composition series of the form

$$\{(1))\} \lhd U_4 \lhd V_4 \lhd A_4 \lhd S_4,$$

where $U_4 = \{(1), (12)(34)\}$ and $V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$ with

$$|S_4/A_4| = 2, |A_4/V_4| = 3, |V_4/U_4| = 2 \text{ and } |U_4/\{(1)\}| = 2$$

which are all primes and so are simple factors!

4 Question 5.

4.1 (b):

Easy calculations show that:

$$\sigma(x_1 + x_2) = x_{\sigma(1)} + x_{\sigma(2)}$$

and then

$$\begin{aligned} G(x_1 + x_2) = \{\sigma(x_1 + x_2) : \sigma \in S_4\} &= \{x_i + x_j : 1 \leq i \neq j \leq n\} \\ &= \{x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_3 + x_4\}. \end{aligned}$$

Also-

$$\begin{aligned} G_{x_1+x_2} &= \{\sigma \in S_4 : \sigma(x_1 + x_2) = (x_1 + x_2)\} = \{\sigma \in S_4 : x_{\sigma(1)} + x_{\sigma(2)} = x_1 + x_2\} \\ &= \{\sigma \in S_4 : \sigma(1) = 1, \sigma(2) = 2 \text{ or } \sigma(1) = 2, \sigma(2) = 1\} \\ &= \{(1), (12)(34), (12), (12)(34)\} \end{aligned}$$

and this is an abelian group of order 4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4.2 (c):

$$\begin{aligned} G(x_1x_2 + x_3x_4) = \{\sigma(x_1x_2 + x_3x_4) : \sigma \in S_4\} &= \{x_{\sigma(1)}x_{\sigma(2)} + x_{\sigma(3)}x_{\sigma(4)} : \sigma \in S_4\} \\ &= \{x_1x_2 + x_3x_4, x_1x_3 + x_2x_4, x_1x_4 + x_2x_3\} \end{aligned}$$

and

$$\begin{aligned} G_{x_1x_2+x_3x_4} &= \{\sigma \in S_4 : \sigma(x_1x_2 + x_3x_4) = x_1x_2 + x_3x_4\} \\ &= \{\sigma \in S_4 : x_{\sigma(1)}x_{\sigma(2)} + x_{\sigma(3)}x_{\sigma(4)} = x_1x_2 + x_3x_4\} \\ &= \{(1), (34), (12), (12)(34), (13)(24), (14)(23), (1324), (1423)\} \end{aligned}$$

ad this is easily seen a group of order 8 isomorphic to D_8, as (34) is of order 2, (1324) is of order 4 and $(34)(1324)(34) = (1423) = (1324)^{-1}$.

4.3 (d):

$$\begin{aligned}G((x_1+x_2)(x_3+x_4)) &= \{\sigma((x_1+x_2)(x_3+x_4)) : \sigma \in S_4\} \\ &= \{(x_{\sigma(1)}+x_{\sigma(2)})(x_{\sigma(3)}+x_{\sigma(4)}) : \sigma \in S_4\} \\ &= \{(x_1+x_2)(x_3+x_4), (x_1+x_3)(x_2+x_4), (x_1+x_4)(x_2+x_3)\}\end{aligned}$$

and

$$\begin{aligned}\sigma((x_1+x_2)(x_3+x_4)) = (x_1+x_2)(x_3+x_4) &\iff (x_{\sigma(1)}+x_{\sigma(2)})(x_{\sigma(3)}+x_{\sigma(4)}) = (x_1+x_2)(x_3+x_4) \\ &\iff \sigma(\{1,2\}) = \{1,2\} \text{ or } \sigma(\{1,2\}) = \{3,4\}\end{aligned}$$

then

$$G_{(x_1+x_2)(x_3+x_4)} = \{(1), (34), (12), (12)(34), (13)(24), (14)(23), (1324), (1423)\} = G_{x_!x_2+x_3x_4}.$$

5 Question 7.

5.1 (a):

It is easy to see that if

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times ... = \prod_{\alpha \in I} \mathbb{Z}_2$$

is a group, where I can be any set of indices, with the property that

$$x \in G \Longrightarrow x = (x_\alpha)_{\alpha \in I}, x_\alpha \in \mathbb{Z}_2 \Longrightarrow x^2 = (x_\alpha^2)_{\alpha \in I} = (0)_{\alpha \in I} = e_G.$$

5.2 (b):

Let $\phi : G \to H$ be an homomorphism. We can define the mappings

$$\pi : G \to G/[G,G], \text{ by } \pi(g) = g \cdot [G,G]$$

is the canonical mapping and

$$j : G/[G,G] \to H, \text{ by } j(g \cdot [G,G]) = \phi(g).$$

It is easily seen that $(j \circ \pi)(g) = j(\pi(g)) = j(g \cdot [G,G]) = \phi(g)$, i.e., that

$$\phi = j \circ \pi.$$

If there are 2 mappings $j_1, j_2 : G/[G,G] \to H$ such that

$$\phi = j_1 \circ \pi = j_2 \circ \pi$$

then for every $g \in G$ we have

$$j_1(g_1 \cdot [G,G]) = j_1(\pi(g)) = \phi(g) = j_2(\pi(g)) = j_2(g \cdot [G,G]),$$

i.e., $j_1 = j_2$ and this follows simply as the mapping π is onto $G/[G,G]$.