Algebraic Structures- Solutions of Homework 8

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1 Question 1.

1.1 (b):

- If $x \in S$ is invertible, it means that there exists $y \in S$ such that $xy = yx = 1_S$. Then, $y \in R$ and $xy = yx = 1_R$ so x is invertible in R.
- The converse is not always true: Take R to be the ring of real numbers and S be its sub-ring that is all the integers. The number 2 has an inverse in $\mathbb{R}$ (which is $1/2$) but 2 is not invertible in $\mathbb{Z}$.

2 Question 2.

2.1 (a):

- If $d \geq 1$: V_d is not a ring, as it is not closed under multiplication: $x^d \in V_d$ but $x^d \cdot x^d = x^{2d} \notin V_d$ as $2d > d$.
- If $d = 0$: V_0 is exactly the field $\mathbb{K}$ and that is a ring.
- if $p_1(x)p_2(x) = 0$ where $p_1, p_2 \in V_d$ then clearly $p_1 = 0$ or $p_2 = 0$, as $p_1$and p_2 are polynomials. Then V_d is an integral domain.

2.2 (c):

- $\mathbb{C}^r(\mathbb{R}, 0)$ is a ring and the only non-obvious properties to check are that $\mathbb{C}^r(\mathbb{R}, 0)$ is closed under summation and multiplication: If $f, g \in \mathbb{C}^r(\mathbb{R}, 0)$ then f, g are continuously differentiable $r-$many times around 0, which imply that so are $f + g$ and fg, so $f + g, fg \in \mathbb{C}^r(\mathbb{R}, 0)$.
- $\mathbb{C}^r(\mathbb{R}, 0)$ is not an integral domain: let

$$f(x) = \begin{cases} x^{r+1} & : x > 0 \\ 0 & : x \leq 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & : x > 0 \\ x^{r+1} & : x \leq 0 \end{cases}$$

then f and g are continuously differentiable $r-$many times in $\mathbb{R}$, clearly not vanishes in $\mathbb{R}$, i.e., $f \neq 0$ and $g \neq 0$, but $f(x)g(x) = 0$ for any $x \in \mathbb{R}$.

2.3 (e):

- This is not a ring, as it is not closed under multiplication: let

$$f(x) = g(x) = \begin{cases} 1/\sqrt{x} & : x \in (0,1) \\ 0 & : x \notin (0,1) \end{cases}$$

then

$$\left| \int_{-\infty}^{\infty} f(x)dx \right| = \left| \int_{-\infty}^{\infty} g(x)dx \right| = \left| \int_{0}^{1} \frac{dx}{\sqrt{x}} \right| < \infty$$

but

$$\left| \int_{-\infty}^{\infty} f(x)g(x)dx \right| = \left| \int_{0}^{1} \frac{dx}{x} \right| = \infty,$$

i.e., $f, g \in R$ but $fg \notin R$.

- This is not an integral domain: let

$$f(x) = \begin{cases} 1 & : x \in (0,1) \\ 0 & : x \notin (0,1) \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & : x \in (2,3) \\ 0 & : x \notin (2,3) \end{cases}$$

clearly $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} g(x)dx = 1$ so f and g are elements in the space, but $f \neq 0, g \neq 0$ and $f(x)g(x) = 0$ for any $x \in \mathbb{R}$.

3 Question 3.

3.1 (b):

Let R be a commutative ring with a unit 1_R and let $I, J \triangleleft R$ such that $I + J = R$.

- If $x \in I \cdot J$, then $x = ij$ for some $i \in I, j \in J$ and as I and J are ideals it follows that $x = ij \in I$ and $x = ij \in J$, so $x \in I \cap j$, i.e., $I \cdot J \subseteq I \cap J$.

- If $x \in I \cap J$, then as $I + J = R$ there exist $i \in I$ and $j \in J$ such that

$$1_R = i + j.$$

Then $x = x \cdot 1_R = x(i + j) = x \cdot i + x \cdot j = i \cdot x + x \cdot j \in I \cdot J$, as $i, x \in I$ and $x, j \in J$, i.e., $I \cap J \subseteq I \cdot J$.

3.2 (c):

Suppose that I is an ideal of $M_{n \times n}(R)$ and let

$$J = \{x \mid \exists M \in I \text{ such that } M_{1,1} = x\}.$$

Recall that $E_{i,j}$ is the matrix with all 0 entries except for 1_R in the (i, j) entry.

1) J **is an ideal of** R**:** If $x, y \in J$ and $r \in R$, there exist $A, B \in I$ such that $x = A_{1,1}$ and $y = B_{1,1}$. As $I \lhd M_{n\times n}(R)$ it follows that

$$Ar, rA, A - B \in I$$

and hence

$$(Ar)_{1,1} = xr \in J, (rA)_{1,1} = rx \in J \text{ and } (A-B)_{1,1} = x - y \in J,$$

which exactly mean that $J \lhd R$.

2) $M_n(J) \subseteq I$**:** If $A \in M_{n\times n}(J)$, then $A = \sum_{i,j=1}^n a_{i,j}E_{i,j}$ where $a_{i,j} \in J$, therefore $(M^{(i,j)})_{1,1} = a_{i,j}$ for some $M^{(i,j)} \in I$ and hence (as $I \lhd M_{n\times n}(R)$):

$$a_{i,j}E_{i,j} = (M^{(i,j)})_{1,1}E_{i,j} = E_{i,1}M^{(i,j)}E_{1,j} \in I$$

for every $1 \leq i, j \leq n$, so $A = \sum_{i,j=1}^n a_{i,j}E_{i,j} \in I$.

3) $I \subseteq M_{n\times n}(J)$**:** If $A \in I$, then for every $1 \leq i, j \leq n$ we have

$$a_{i,j} = (E_{1,i}AE_{j,1})_{1,1}$$

and as $I \lhd M_{n\times n}(R)$, $E_{1,i}AE_{j,1} \in I$ and thus $a_{i,j} \in J$, i.e., $A \in M_{n\times n}(J)$.

4 Question 5.

4.1 (b):

$\mathbb{H}^\times$ is the subset of $\mathbb{H}$ consists of all invertible elements in $\mathbb{H}$, so we need to show that every element $q \in \mathbb{H}$ that is $q \neq 0$, is invertible, or in other words that $\mathbb{H}$ is a division ring: let $q \neq 0$ in $\mathbb{H}$, then $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ for some $a, b, c, d \in \mathbb{R}$ such that a, b, c, d are not all equal 0, i.e., such that $a^2 + b^2 + c^2 + d^2 \neq 0$. It is a simple computation to show that

$$q \cdot \overline{q} = (a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}) = a^2 + b^2 + c^2 + d^2$$

and therefore q is invertible in $\mathbb{H}$ and

$$q^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}\overline{q} = \frac{a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}}{a^2 + b^2 + c^2 + d^2} \in \mathbb{H}.$$

4.2 (c):

Define $\|q\| := \sqrt{q \cdot \overline{q}}$ for every $q \in \mathbb{H}$, that can be written as

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}, \quad \text{when } q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

This defined a norm on $\mathbb{H}$:

- For every $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$:

$$\|q\| = 0 \iff \sqrt{a^2 + b^2 + c^2 + d^2} = 0 \iff a = b = c = d = 0 \iff q = 0.$$

- For every $\alpha \in \mathbb{R}$:

  $$\|\alpha q\| = \|\alpha a + \alpha b\mathbf{i} + \alpha c\mathbf{j} + \alpha d\mathbf{k}\| = \sqrt{(\alpha a)^2 + (\alpha b)^2 + (\alpha c)^2 + (\alpha d)^2} = |\alpha| \cdot \|q\|.$$

- For every $q_1 = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}, q_2 = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k} \in \mathbb{H}$:

  $$\begin{aligned}\|q_1 + q_2\| &= \sqrt{(a_1+a_2)^2 + (b_1+b_2)^2 + (c_1+c_2)^2 + (d_1+d_2)^2} \\ &= \left\| \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \right\|_{\mathbb{R}^4} \leq \left\| \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \right\|_{\mathbb{R}^4} + \left\| \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \right\|_{\mathbb{R}^4} \\ &= \sqrt{a_1^2 + b_1^2 + c_1^2 + d_1^2} + \sqrt{a_2^2 + b_2^2 + c_2^2 + d_2^2} = \|q_1\| + \|q_2\|\end{aligned}$$

  where we use a property of the standard norm of $\mathbb{R}^4$.

So $\|\cdot\|$ defines a norm on $\mathbb{H}$ and for every $q_1, q_2 \in \mathbb{H}$:

$$\begin{aligned}q_1 \cdot q_2 &= (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1c\mathbf{k})(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i} \\ &+ (a_1c_2 + c_1a_2 - b_1d_2 + d_1b_2)\mathbf{j} + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\mathbf{k}\end{aligned}$$

so-

$$\begin{aligned}&\|q_1 \cdot q_2\|^2 = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)^2 \\ &+ (a_1c_2 + c_1a_2 - b_1d_2 + d_1b_2)^2 + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)^2 \\ &= a_1^2a_2^2 - 2a_1a_2b_1b_2 - 2a_1a_2c_1c_2 - 2a_1a_2d_1d_2 + b_1^2b_2^2 + 2b_1b_2c_1c_2 + 2b_1b_2d_1d_2 + c_1^2c_2^2 + 2c_1c_2d_1d_2 + d_1^2d_2^2 \\ &+ a_1^2b_2^2 + 2a_1b_2b_1a_2 + 2a_1b_2c_1d_2 - 2a_1b_2d_1c_2 + b_1^2a_2^2 + 2b_1a_2c_1d_2 - 2b_1a_2d_1c_2 + c_1^2d_2^2 - 2c_1d_2d_1c_2 + d_1^2c_2^2 \\ &+ a_1^2c_2^2 + 2a_1c_2c_1a_2 - 2a_1c_2b_1d_2 + 2a_1c_2d_1b_2 + c_1^2a_2^2 - 2c_1a_2b_1d_2 + 2c_1a_2d_1b_2 + b_1^2d_2^2 - 2b_1d_2d_1b_2 + d_1^2b_2^2 \\ &+ a_1^2d_2^2 + 2a_1d_2d_1a_2 + 2a_1d_2b_1c_2 - 2a_1d_2c_1b_2 + d_1^2a_2^2 + 2d_1a_2b_1c_2 - 2d_1a_2c_1b_2 + b_1^2c_2^2 - 2b_1c_2c_1b_2 + c_1^2b_2^2 \\ &= (a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2) = \|q_1\|^2 \cdot \|q_2\|^2.\end{aligned}$$

4.3 (f):

Let $q_1 = q_2 = q$ and use the calculation of $q_1 \cdot q_2$ above to show that

$$q^2 = (a_1^2 - b_1^2 - c_1^2 - d_1^2) + (2a_1b_1)\mathbf{i} + (2a_1c_1)\mathbf{j} + (2a_1d_1)\mathbf{k}$$

and thus

$$q^2 = -1 \iff a_1^2 - b_1^2 - c_1^2 - d_1^2 = -1, a_1b_1 = a_1c_1 = a_1d_1 = 0.$$

If $a_1 \neq 0$, then $b_1 = c_1 = d_1 = 0$ and $\mathrm{a}_1^2 = -1$ and this is a contradiction, as $a_1 \in \mathbb{R}$, so

$$\{q \in \mathbb{H} : q^2 = -1\} = \{q = b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k} : b_1^2 + c_1^2 + d_1^2 = 1\}$$

and there infinitely many solutions.

5 Question 6.

If $x \in R$ such that $x^2 = 1_R$, then $0 = x^2 - 1_R = (x - 1_R)(x + 1_R)$ and since R is an integral domain (has no zero divisors) it follows that

$$x - 1_R = 0 \text{ or } x + 1_R = 0,$$

i.e., that $x = 1_R$ or $x = -1_R$.